USING

INTEREST RATE DERIVATIVES

IN TRADING AND MANAGING RISK

PRE-COURSE READING

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CONTENTS

Background Information on Money Market Instruments

The following material contains basic information about money market instruments that will form part of the course. You will probably be familiar with some of this already. The course will deal with how these instruments are used and valued and you may need to refer to the definitions in this material during lectures and workshops.

Cash Markets

The interbank market for short-term deposits and borrowings out of today (or Spot) is often referred to as the Cash Market. This name is most commonly used to differentiate this market from that for futures and FRAs. The market is highly liquid for periods of up to one year with the three and six month periods being the most popular.

Cash Calculation Basis

Rates are quoted on a simple interest basis with interest being paid at the end of the period for which it is calculated. The number of days in a year for interest calculation purposes varies between currencies. The majority of currencies including the Euro and the US Dollar have a 360-day year (Actual/360 basis) while Sterling, some of the old commonwealth currencies and domestic Yen use 365 days (Actual/365 basis).

For example £100 put on deposit for 180 days at 10% will earn:

$$\text{£}100 * \frac{10}{100} * \frac{180}{365} = \text{£}4.94$$

While $100 put on deposit for the same time period at the same rate earns:

$$\$100 * \frac{10}{100} * \frac{180}{360} = \$5.00$$

The other basis that we shall use a lot in the course is 30/360. This is the norm in the Eurobond and US corporate bond markets, but not the cash markets. Under this regime, each month has exactly 30 days.

For currencies other than Sterling a deposit or borrowing rate agreed today would normally relate to a transaction starting at Spot (in two days time). Sterling transactions are usually out of today or tomorrow.

Deposits can also be made for short periods for example:

Overnight:	From today to the next working day.
Tom-Next:	Between one and two working days ahead.
Spot-Next:	Between two and three working days ahead.
Spot-Week:	Between two working days and two working days plus one calendar week ahead.

Cash Day Count

When periods are quoted a “month” normally refers to a calendar month so one month from 24 September will run until 24 October. However, if 24 October is a non-working day then the period end will be rolled forward to the following working day. The only exception to this is when rolling the date forward takes you into a new month. Under this circumstance the period end is rolled backward to the previous working day. So if 29 and 30 November were Saturday and Sunday then one month from 30 October would be 28 November.

This rule is known as the modified following business day (or banking day) convention. Other markets, such as those for bonds, apply different rules. “Modified following” is the most common rule for the money market instruments and we will stick to using this unless otherwise specified.

Short Term Interest Rate Futures

Because the market for financial futures developed from that for commodity futures the structure of these instruments and their markets are very different from their "OTC" counterparts such as cash and FRAs. (OTC stands for "Over The Counter")

Unlike the market for cash, swaps or FRAs, futures markets have the following characteristics:

- Futures markets are focused around organised exchanges. Members join the exchanges through the purchase of a "seat" which carries the right to trade.
- In the past the exchanges provided a physical location where trading between members was accomplished by "open outcry" (face-to-face). Open outcry trading had to be executed in designated areas, usually areas called "pits". However, apart from some notable exceptions such as the Eurodollar contract in Chicago trading has now transferred to electronic platforms that allow trades to be executed anonymously and very rapidly.
- The contracts traded are basically rights to make or take delivery of a given instrument on the delivery dates. The fact that actual delivery will not occur in many cases does not prevent them from being freely traded. Non-deliverable contracts are closed out at a specific price at the end of their life.
- The contracts are standardised as to the amount and type of the underlying financial "commodity". The contract expiry dates are also fixed.
- For an individual to trade futures contracts on the exchange, a sum of money called margin must be posted with a "Clearing Member" of the exchange. This margin is transferred to the exchange "Clearing House".
- At the end of each day positions that have been taken in the futures market are valued against the daily settlement price. This price is established in the closing minutes of trading. If the position has earned a profit, funds will be transferred into the position holder's margin account. If the position has recorded a loss, then there may be a call for additional funds to be deposited in the margin account.
- On some exchanges limits on price movement exist to protect all participants from temporary, erratic price behaviour, which could cause unusual losses.

In futures markets there are a number of specific entities and individuals that perform well-defined roles. The main ones are as follows:

Futures Exchange	This is the organisation that used to provide the physical facilities for trading and still does for some contracts. It also sets down the rules and regulations on every aspect of market activity. The exchange issues a given number of seats or trading permits, which can be purchased by individuals or corporations acceptable under the terms of membership. Ownership of a seat confers the right to trade. Day to day operations are monitored through a number of committees which consist of members of the exchange. The exchange is also responsible for the arbitration of disputes, which is accomplished through either a committee or an exchange staff ruling.
Clearing House	This is attached to the exchange, as either a department or as a separate corporation with an agreement to provide clearing services to the exchange. The Clearing House performs several important functions. It is responsible for the transfer of funds associated with the margin deposits and the daily margin calls. For deliverable contracts it handles the delivery process to ensure that a buyer presents funds and a seller presents the commodity before delivery takes place. To keep track of the huge amount of information involved requires enormous computing and data processing capabilities. One of the most important functions performed by the Clearing House is to stand as the counterparty to every transaction. When a buyer and seller of a given contract agree on a price, that trade is registered with the Clearing House, which then becomes counterparty to both buyer and seller. This means that either party can come to the market to close their position without referring to the other. When the buyer wants to sell his contract and close his position, he can do so because the Clearing House has recorded his previous transactions and will offset the buy and sell orders. By being counterparty to every trade, the Clearing House removes a trader's need to consider the credit worthiness of the other side of a transaction. The credit standing of the Clearing House itself is of course a principal concern. Since all trades must be registered with the Clearing House, a system of Clearing Members exists. These are members of the exchange that have qualified as Clearing Members by being able to provide guarantees to certify their financial ability to handle the clearing process. The Clearing Members' function is to provide a route for the registration of trades and the posting of margin. Non-Clearing Members pass details of their trades to a Clearing Member on a daily basis. Along with this goes a transfer of the required margins payments. The Clearing Member is then responsible for the registration of all trades and the posting of margin amounts with the Clearing House.

Electronic Trading	In the electronic market, participants are anonymous and orders are matched automatically according to their price and time of entry. This has generally led to increased participation, competition and an improvement in bid/offer spreads and liquidity. The electronic market is operated by a "Trading Host" which performs order matching and trade and price reporting. In order to participate in the market, members require a trading application that connects to the Trading Host. These may be integrated with other trading systems to allow rapid execution of complex trades.
Futures Contract Details	A typical short-term interest rate future is the Three-Month Eurodollar Future. The contract details are given as follows:

Contract:	Three-Month Eurodollar Interest Rate Future.
Unit of Trading:	$1,000,000.00
Delivery Months:	March, June, September, and December and serial months
Delivery Day:	First business day after the last trading day
Last Trading Day	11.00 Two days before the third Wednesday of the delivery month.
Quotation:	100 minus rate of interest
Minimum Price Move:	0.0025 ($6.25) this is a quarter of a tick

The price of the contract is quoted as 100 minus an interest rate but, the amount of money changing hands as the result of a given move in the price is derived from the contract amount and the period of the contract.

A move of one tick (0.01%) is worth $1,000,000 * 90/360 * 0.0001 = $25.

Contracts can be bought and sold for anything up to several years before the last trading date. For the three month Eurodollar contract, this is nominally ten years. The interest rate that is reflected in the price is therefore related to the forward rate for a period of 90 days from the delivery date. It is however not exactly the forward rate for reasons of convexity that will be explained on the course.

Because the contracts are for three-month periods, the whole year is spanned by four delivery months. This is apart from a few gaps or overlaps between the end of the deposit period for one contract and the delivery date of the next.

Forward Rate Agreements (FRAs)

FRAs or Forward Rate Agreements are instruments designed to provide a hedge against the level of short-term interest rates at some period in the future. Formally, they are agreements between two parties, one of whom is usually a bank, to pay or receive a sum of money. This sum is based on a defined principal amount and the difference between an interest rate set at the date of agreement (the FRA rate) and a specific cash rate, normally LIBOR, at some time in the future.

For example the buyer of 6-9 FRA will receive compensation from the seller if the three-month LIBOR rate in six months time rises above the rate at which he bought the FRA. Conversely he will pay compensation to the seller if LIBOR drops below the FRA rate.

If the buyer of this FRA is a treasurer who has to roll-over a three-month loan in six months time at LIBOR he will have locked in his borrowing rate for that period as long as the principal amount of the loan matches that of the FRA. This is because as his cost of funds rises his receipts from the FRA exactly compensate him for the increased payments on the loan. Conversely if rates and his cost of borrowing drops he will be obliged to pay under the FRA and those payments will bring his cost of funds back to the FRA rate.

The exact formula for calculating the payment made under the FRA is:

$$C = \frac{P * (L - R_{FRA}) * \frac{D}{360}}{1 + (L * \frac{D}{360})}$$

P	=	Principal Amount
C	=	The compensation payment made by the seller to the buyer of the FRA on the settlement date at the beginning of the LIBOR period.
L	=	The LIBOR rate on which the FRA is based.
R_{FRA}	=	The FRA rate agreed at the time of purchase.
D	=	The number of days in the LIBOR period.

Traditional FRAs are settled at the beginning of the LIBOR period on the "Settlement Date". The payment amount depends on the level of the appropriate LIBOR rate on the "Fixing Date". The Fixing date is normally two days before the Settlement date for those currencies, which trade out of Spot. For currencies like sterling, which trade out of today, the two dates are the same.

Futures v FRAs

FRAs have sometimes been described as "Over The Counter" futures contracts and this is in part true. The fundamental differences between futures and FRAs are as follows:

- There are no specific dates for FRAs. The most liquid periods are the three and six month "runs" (3-6, 6-9, 3-9, 6-12 etc.).
- FRAs are quoted in terms of interest rates.
- Day count and interest calculations are usually done on the same basis as the cash deposit markets in the currency of the FRA. Three months is therefore not always 90 days.
- FRAs are not traded through an organised exchange. Parties must therefore make their own assessment of each other's credit.
- There are no margin requirements for FRAs.
- There are no fixed contract amounts for FRAs, but in practice prices are generally quoted for a minimum principal of $25,000,000.00 or equivalent.

The major benefits of using FRAs rather than futures are:

- Reduced administration as the result of not having to cope with initial and daily margin calls.
- The cashflow and funding aspects of margin payments are eliminated.
- The flexibility of FRA dates and amounts allow them to be used as a much more accurate hedging vehicle.

The major benefits of using futures are:

- The bid-offer spread is narrower than for FRAs particularly near to delivery.
- The Liquidity of the futures market can be much greater than the FRA market in specific currencies and maturities. For example, trades of 2000 contracts in the Three-Month Eurodollar future are not unusual, but doing a two billion Dollar FRA is trickier, but not impossible.

Interest Rate Swaps	These instruments will be covered in depth on the course. At the moment it is necessary to be aware of some basic definitions.

An interest rate swap agreement is a contract whereby two parties agree to exchange payments equivalent to the interest payable on some notional principal amount. Payments are based on two different indices, normally one fixed and one floating. The fixed rate payer makes periodic payments to the other party corresponding to a fixed interest rate and based on the notional principal. The floating rate payer makes a series of payments based on a floating rate such as three month LIBOR and the notional principal.

An interest rate swap contract is defined by various parameters:

Term:	The swap start and end dates define the term of the contract. Contracts can be for anything from a few months to many years. Frequently traded maturities range from one to forty years. Forward starting deals are not unusual.
Notional Principal:	The principal amount from which the cashflows are calculated is not normally exchanged in an interest rate swap. Typical market amounts range from $25,000,000.00 upward.
Fixed Rate:	The rate against which fixed side payments are calculated.
Floating Rate:	The rate against which floating side payments are calculated. This can be a LIBOR, Treasury Bill or other rate such as Prime.
Rate Fixing Dates:	The series of dates on which the floating rate is determined (Reset Dates).
Payment Dates:	The series of dates on which payments are made on the fixed and floating sides of the swap.
Reset Interval:	The frequency with which the floating rate is recalculated. This is normally one, three, six or twelve months. It can also be tied to the delivery dates of the relevant three-month interest rate futures contracts.
Stub Period:	A period at the start or end of the swap which is longer or shorter that the period between reset dates for that side of the swap.

Options on Interest Rates

Before the course it is important to gain a general feel for the factors that affect the value of option contracts. For this reason, the material below does not specifically mention interest rate options until the end.

In general terms, an option is the right but not the obligation to buy or sell a given amount of a specific commodity at a certain price on or before a predetermined date. In order to gain this right the buyer of the option will pay a sum of money to the seller.

In this description the word "commodity" refers to a wide range of goods and transactions. It includes soya beans, property and financial instruments such as forward interest rates, FRAs and swaps.

Definitions:

Call:	The right to buy
Put:	The right to sell
Holder:	The buyer of the option
Writer:	The seller of the option
Premium:	The sum paid by the Holder to the Writer for the option.
Exercise Price:	The price at which the commodity can be sold or bought by the Holder.
Contract Amount:	The amount of commodity bought or sold when the option is exercised.
Expiry Date:	The date on or before which the option Holder may exercise the option.
American Style:	An option under which the Holder has the right to exercise on any date up to and including the expiry date.
European Style:	An option under which the Holder has the right to exercise on the expiry date.
Path Dependant:	An option that has a strike price that depends on the price history of the commodity. Included in this class are: Look Back Options, Average Rate Options and Down and Out Options.
Compound:	An option on an option.

Market Terms:

In the Money:	A call option is in the money when the market price of the commodity is greater than the option strike price. A put option is in the money when the market price is less than the strike price.
Out of the Money:	A put option is out of the money when the market price of the commodity is greater than the option strike price. A call option is out of the money when the market price is less than the strike price.
At the Money:	Both put and call options are at the money when the market price of the commodity is equal to the option strike price.
OTC:	An option traded "Over The Counter" between two parties directly and not through an exchange.
Exchange Traded:	An option traded on the floor of a recognised exchange.
Intrinsic Value:	The element of the option value that is made up of the present value of the profit, if any, from immediate exercise.
Time Value:	Option value minus intrinsic value.
Delta:	The rate of change of the option value with respect to the commodity price.
Gamma:	The rate of change of the option Delta with respect to the commodity price.
Rho:	The rate of change of the option value with respect to the risk-free interest rate for the period between today and expiry.
Vega (Lambda)	The rate of change of the option value with respect to the volatility of the commodity price.
Theta:	The rate of change of the option value with respect to the time remaining to expiry.

Option Pricing:

The key factors that affect the value of an option are:

- The commodity price
- The volatility of the commodity price
- The risk free rate of interest for the period to maturity
- The option strike price
- The maturity date

An option can never be worth less than zero. Even if an option is a long way out of the money there is still a chance that it might become in the money before expiry and this chance has a positive value. A call option can never be worth more than the value of the contract amount. If a call option was worth more than the contract amount of the underlying commodity then there would be an immediate arbitrage by selling the call and buying the commodity.

Figure 1

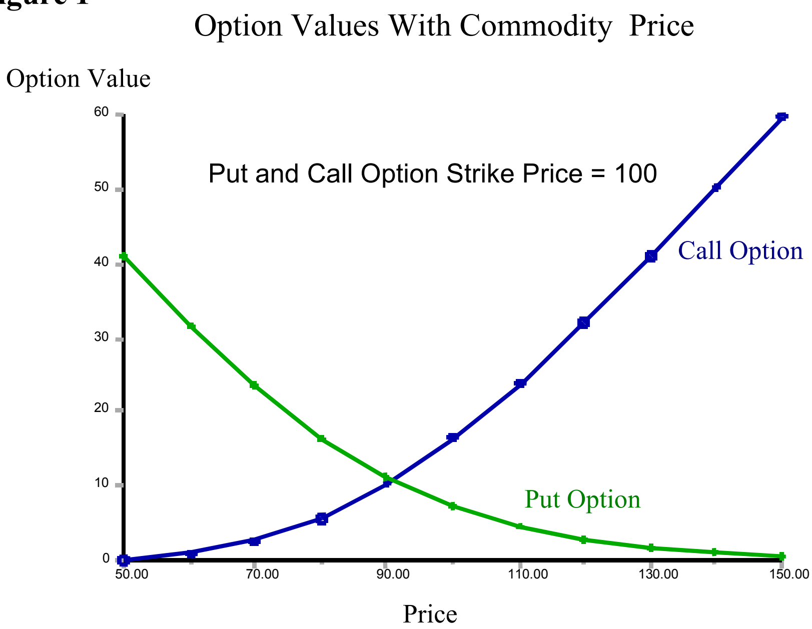


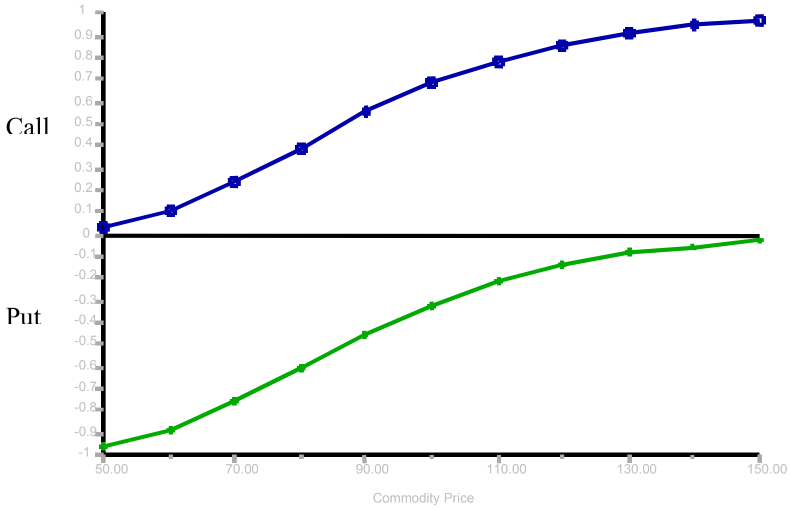
Figure 1 illustrates that the value of a call option increases with increasing speed as the commodity price gets closer to the exercise price and converges to increasing at the same rate as the commodity price as the option becomes further in the money. The converse of this is true of a put option.

Hedging Factors

The rate of change of the option value with respect to the commodity price (the option Delta) is shown by the slope of the value curve. This is illustrated for European put and call options in Figure 2 below.

Figure 2

Option Deltas v Commodity Price



Figures 3 and 4 illustrate the way that Gamma and Theta for European style options change with commodity price.

Figure 3

Option Gamma With Commodity Price

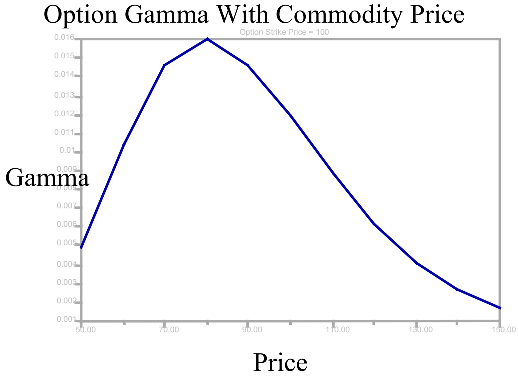
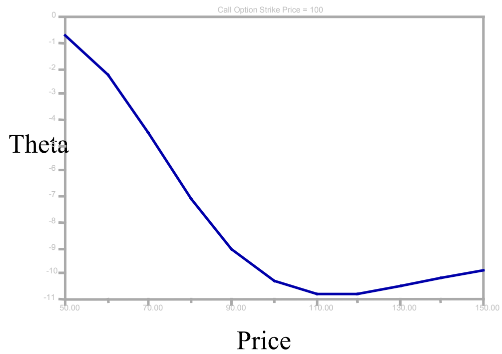


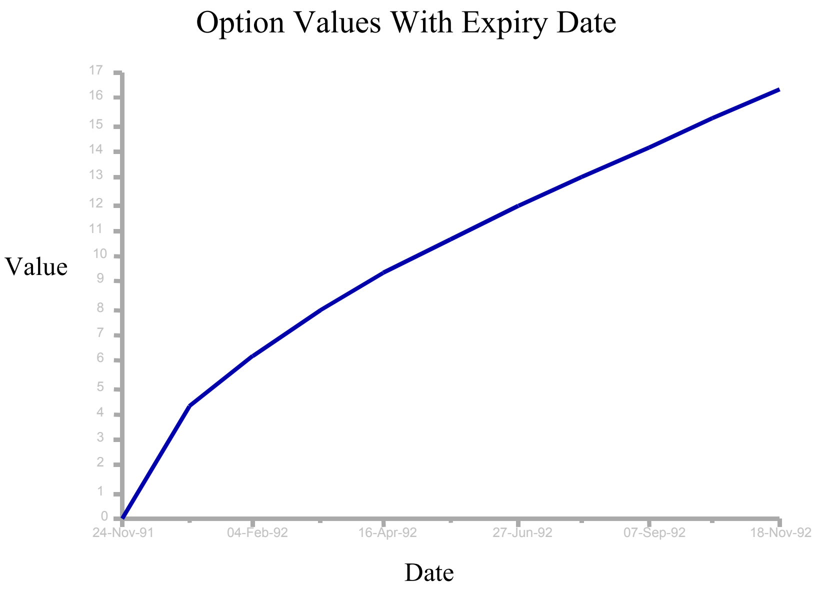
Figure 4

Option Theta With Commodity Price



The way in which a European style call option's value decreases with time to expiry is illustrated in Figure 5.

Figure 5



Figures 6 and 7 show the effect of changes in the volatility of the commodity price on the value of European style call and put options.

Figure 6

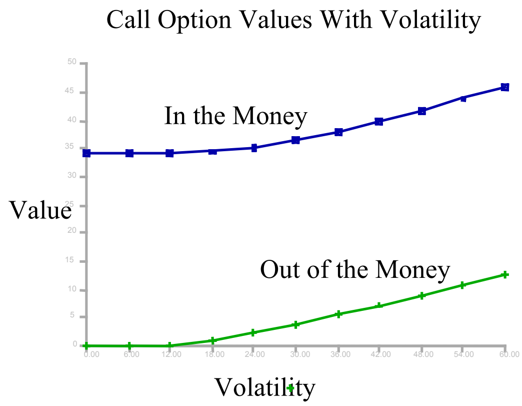
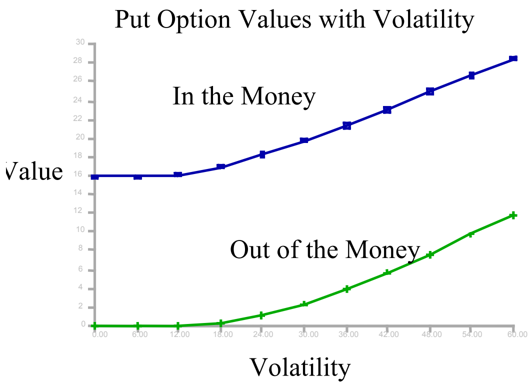
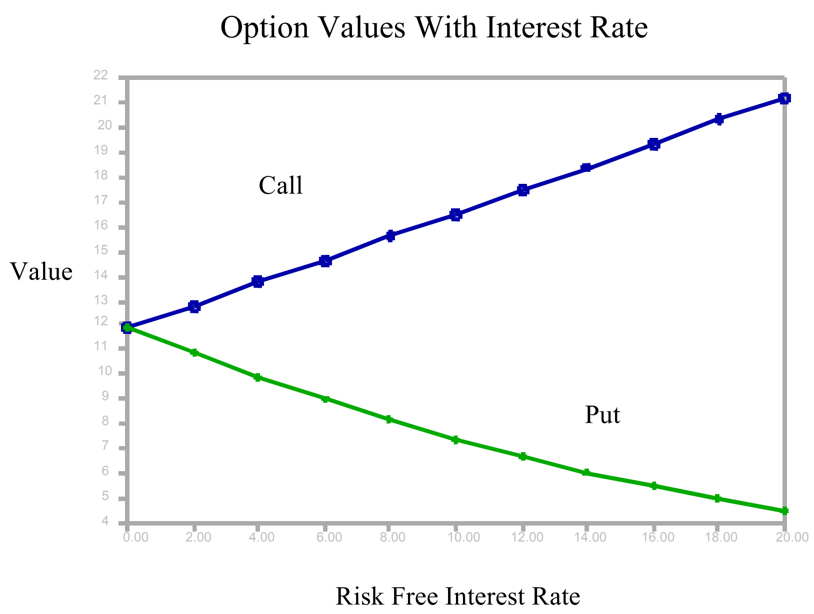


Figure 7



The risk free rate of interest is also an important factor in determining the price of all options regardless of the nature of the commodity. Figure 8 illustrates how changes in the risk free rate can alter the value for European style put and call options.

Figure 8



Options on Interest Rates (Caps Floors, Collars and IRGs)

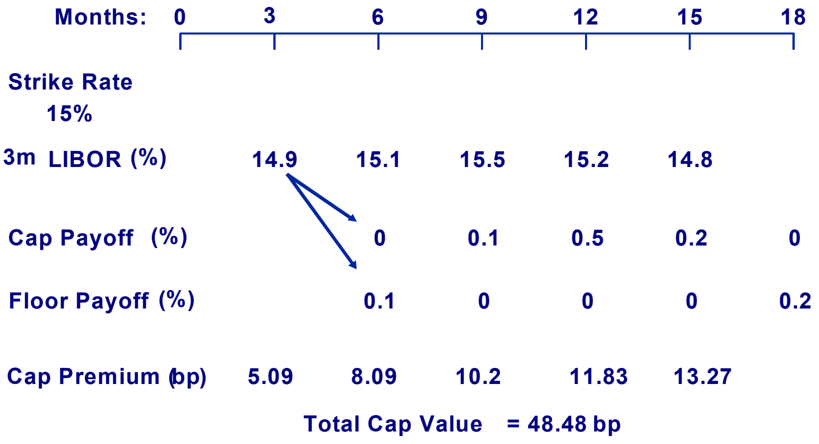
The basics of option pricing apply to these instruments as it does to other options. With these instruments the commodity is a forward interest rate and the strike price is expressed as a percentage. They are generally European style options and the strike rate applies to a specific period in the future. The commodity price is the current forward rate for that period and the contract amount is expressed as a notional principal.

For example under the terms of a standard cap the buyer receives the right to compensation if an agreed floating rate (often 3 month LIBOR) is above the cap strike rate on a specific series of days. The amount of compensation is calculated from the tenor of the floating rate, a principal amount, and the difference between the floating rate and the strike rate. If the floating index used is below the strike rate on the rate fixing date then the buyer of the cap receives no payment.

While the buyer has the right to receive a payment under the terms of the cap the seller has the obligation to make the payment. The seller will therefore receive a premium from the buyer for taking on the risk of having to pay the buyer over the life of the instrument.

A floor operates in a very similar way to a cap except that the buyer will receive payment if the floating index is lower than the strike rate on the rate fixing dates. Figure 9 shows a typical series of payments on an 18-month cap and floor against 3 month LIBOR.

Figure 9



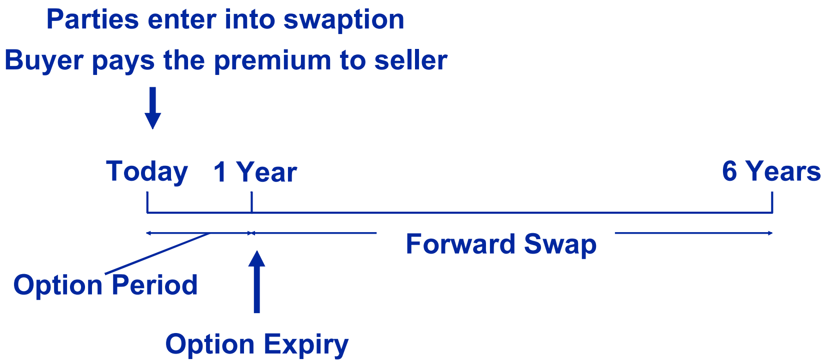
Interest Rate Options (Swaptions)

A swaption is an option on a swap. The instrument comes in two basic types a payer and a receiver swaption. The holder of a payer swaption has the right, but not the obligation to enter into an interest rate swap where he is making fixed rate payments in exchange for receipts based on a floating interest rate. The holder of a receiver swaption has the right to enter into an interest rate swap where he is receiving fixed. Conversely the seller of a swaption has an obligation to enter into a swap as counterparty to the swaption holder. As with the majority of options the holder pays the seller a premium when he takes out the option.

The swaption strike rate is the rate at which the option holder will be making or receiving fixed rate payments if the option is exercised. The floating rate on the majority of swaptions is three, six or twelve month LIBOR.

The most common swaption structures are relatively short-term European style options to enter into longer-term interest rate swaps. The example below illustrates a one-year option into a five-year swap. This is a very typical market structure. Normally the swap starts when the option expires so that the tenor of this swaption is six years.

Figure 10



In the example the option was European and could only be exercised at expiry. It is possible to buy, but quite difficult to value, American style swaptions. There the option can be exercised into a swap of a specific maturity at any time up until the expiry date.

The majority of swaptions are not actually exercised into swaps but are settled for cash. If, in our example, the swaption strike rate is 8.5% and at expiry the five-year swap rate is 8% then the holder of a receiver swaption will get a cash payment. A payer swaption will expire worthless. A receiver swaption is considered to be "in the money" if the forward swap fixed rate is less than the swaption strike rate. The reverse is true for a payer swaption.

As we can see the valuation of a standard European style swaption falls into two parts: The valuation of a swap and the valuation of the option on that swap. The swap to be valued runs from the end of the option period to the final maturity date of the swaption. In the example shown above this would be a forward start swap beginning in one year and ending in six years time.

In order for the present value of the option to be worked out the rate at which it is possible to enter into an appropriate forward start interest rate swap today must be calculated. It is this break-even swap rate that goes into the option-pricing model.

The type of option pricing model that is used for valuing the option element of the swaption is the same as that for caps and floors. The difference is that in the case of swaptions it is the forward swap price not forward LIBOR that is the underlying instrument and for swaptions there is only one option to evaluate.

The key factors for valuing the option are:

- The underlying instrument price (forward swap rate).
- The volatility of the forward swap rate.
- The strike rate.
- The yield curve.
- The maturity of the option.

Swaption Pricing

The approach to pricing European style swaptions is very similar to that used for pricing caps and floors.

The price of a cap or floor is the sum of the prices of a series of short-term options on forward LIBOR. There the underlying instrument against which the option is priced is the forward interest rate.

Swaptions contain only a single option on a forward swap. So in that case the underlying instrument is the forward swap rate. Because of this we can use Black's model to value European style options on interest rate swaps.

As with caps and floors the assumptions underlying the model are:

- There is no default risk inherent in the underlying forward swap.
- There are no transaction costs or liquidity constraints in the forward swap market.
- There is no cash impact of entering into a forward swap.
- The underlying asset changes through time following the diffusion process shown below.

$$\frac{dS}{S} = \mu dt + \sigma dz$$

Where:

S	=	The forward swap rate at a given time.
m	=	A drift term
dz	=	An increment of standard Brownian motion
s	=	The volatility of the forward swap price (a constant).

This means that, like forward LIBOR in the caps model, we assume that the forward swap rate is lognormally distributed.

As a result we can use the standard result from Black's model and say:

$$C(S,t) = SN(d_1) - EN(d_2)$$

Where:

C	=	The value of a European swaption
N(.)	=	The cumulative standard normal distribution and:
E	=	The swaption exercise price.
T_1	=	Start date of the forward swap.
t	=	The valuation date.

$$d_1 = \frac{\text{Log}\frac{S}{E}}{\sigma\sqrt{T_1 - t}} + \frac{1}{2}\sigma\sqrt{T_1 - t}$$

$$d_2 = d_1 - \sigma\sqrt{T_1 - t}$$

DEAN WITTER
INSTITUTIONAL FUTURES
September 16, 19

The Convexity Bias in Eurodollar Futures

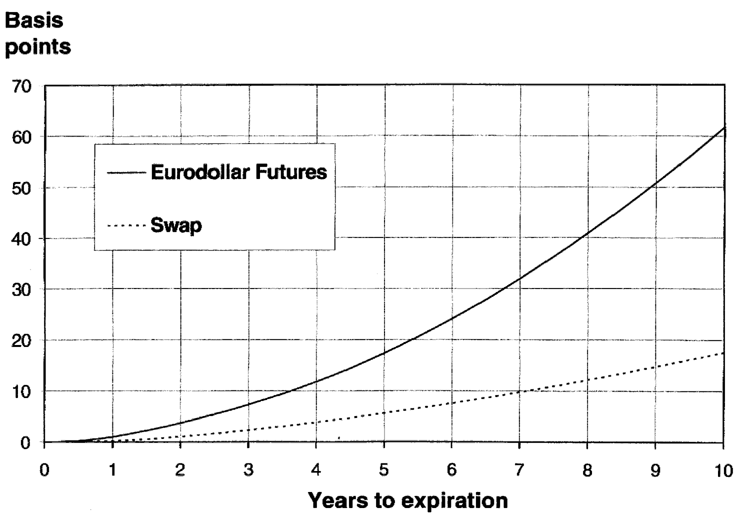
research note

Galen Burghardt • Bill Hoskins

There is a systematic advantage to being short Eurodollar futures relative to deposits, swaps, or FRAs. Because of this advantage, which we characterize as a convexity bias, Eurodollar futures prices should be lower than their so-called fair values. Put differently, the 3-month interest rates implied by Eurodollar futures prices should be higher than the 3-month forward rates to which they are tied.

The bias can be huge. As the chart shows, the bias is worth little or nothing for futures that have less than two years to expiration. For a futures contract with 5 years to expiration, however, the bias is worth about 17 basis points. And for a contract with 10 years to expiration, the bias can easily be worth 60 basis points.

Convexity Bias



The presence of this bias has profound implications for pricing derivatives off the Eurodollar futures curve. For example, a 5-year swap yield should be about 6 basis points lower than the yield implied by the first 5 years of Eurodollar futures. A 10-year swap yield should be about 18 basis points lower. And the differential for a 5-year swap 5 years forward should be around 36 basis points. (These estimates are explained in Exhibits 9, 13, and 14.)

These are big numbers. A 6 basis point spread is worth more than $200,000 on a $100 million 5-year swap. An 18 basis point spread is worth about $1.2 million on a $100 million 10-year swap.

Although the swaps market has begun to recognize this problem, swap yields still seem too high relative to those implied by Eurodollar futures rates. (See Exhibit 15.) If so, then there is still a substantial advantage in favor of being short swaps and hedging them with short Eurodollar futures. Also, because the value of the convexity bias depends so much on the market's perceptions of Eurodollar rate volatilities, one should be able to trade the value of the swaps/Eurodollar rate spread against options on forward Eurodollar rates. The convexity bias also affects the behavior of the yield spreads between Treasury notes and Eurodollar strips.

Students of Eurodollar futures pricing should like this note. The standard approach to estimating the value of the convexity bias (also known as the financing bias) has been bound up in complex yield curve simulations and option pricing calculus. And, although such methods can yield reasonable enough answers, we show how the problem can be solved much more simply. For that matter, anyone armed with a spreadsheet program and an understanding of rate volatilities and their correlations can estimate the value of the convexity bias without recourse to expensive research facilities.

I would like to take this occasion to welcome Bill Hoskins to our futures research group. Bill has just completed his Ph.D. work in finance at the University of Chicago and promises to make some very valuable contributions to our understanding of pricing and hedging with futures.

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The difference between a futures contract and a forward contract is more pronounced for Eurodollar futures, swaps, and FRAs than for any other commodity. In particular, there is a systematic bias in favor of short Eurodollar futures relative to deposits, swaps, or FRAs. As we show, the value of this bias is particularly large for futures contracts with expirations ranging from five to ten years. The purpose of this note is to show

- why the difference is so important for Eurodollar futures
- how to estimate the value of the difference
- what traders can do about the difference

What we find is that the implications for swaps traders and those who manage swaps books are particularly important. Given the rate volatilities that we have observed over the past four years or so, it seems that market swap yields should be several basis points lower than the implied swap yields that one calculates from the rates implied by Eurodollar futures prices. Judging by current spreads between these rates, it appears that the swaps market has not fully absorbed the implications of this pricing problem. As a result, there still appear to be profitable opportunities for running a book of short swaps hedged with short Eurodollar futures. By the same token, this pricing problem raises serious questions about how a swaps book should be marked to market.

INTEREST RATE SWAPS AND EURODOLLAR FUTURES

Interest rate swaps and Eurodollar futures both are driven by the same kinds of forward interest rates. But the two derivatives are fundamentally different in one key respect. With an interest rate swap, cash changes hands only once for each leg of a swap and then only in arrears. With a Eurodollar futures contract, gains and losses are settled every day. As it happens, the difference in the way gains and losses are settled affects the values of swaps and Eurodollar futures relative to one another. In particular, there is a systematic bias in favor of a short swap (that is, receiving fixed and paying floating) and against a long Eurodollar futures contract. Or, one can think of the short Eurodollar position as having an advantage over a long swap. Either way, because swap prices are so closely tied to Eurodollar futures prices, it is important to know how much this bias is worth.

The easiest way to understand the difference between the two derivatives is through a concrete example that compares the profits and losses on a forward swap with the profits and losses on a Eurodollar futures contract.

A forward swap

A plain vanilla interest rate swap is simply an arrangement under which one side agrees to pay a fixed rate and receive a variable or floating rate over the life of the swap. The other side agrees to pay floating and receive fixed. The amounts of money that one side pays the other are determined by applying the two interest rates to the swap's notional principal amount.

The typical swap allows the floating rate to be reset several times over the swap's life. For example, a 5-year swap keyed to 3-month LIBOR would require the value of the floating rate to be set or reset 20 times—once when the swap is transacted and every three months thereafter. One can think of the swap, then, as having 20 separate segments with the value of each segment depending on the swap's fixed rate and on the market's expectation today of what the floating rate will be on that segment's rate setting date.

The starting point for our example is the structure of Eurodollar futures prices and rates shown in Exhibit 1. These were the final settlement or closing prices on Monday, June 13, 1994. Each of the implied futures rates roughly corresponds to a three-month period. The actual number of days covered by each of the futures contracts is shown in the right hand column.

Now consider a swap that settles to the difference between a fixed rate and the value of 3-month LIBOR on March 15, 1999. On June 13, 1994, this would be a forward swap whose rate setting date is 4-3/4 years away and whose cash settlement

Exhibit 1

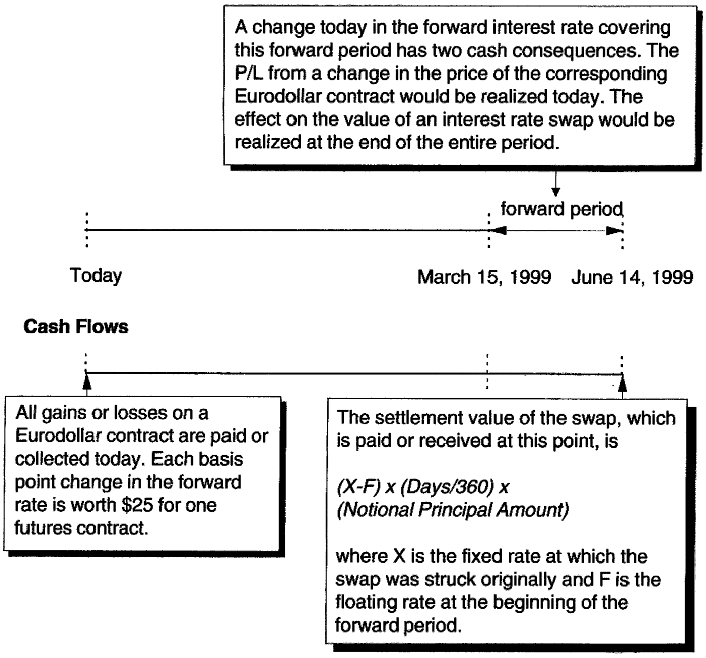
Structure of Eurodollar Futures Rates
(June 13, 1994)

	Eurodollar futures			
Quarter	Expiration	Price	Implied futures rate (percent)	Days in period
1	6/13/94	95.44	4.56	98
2	9/19/94	94.84	5.16	91
3	12/19/94	94.14	5.86	84
4	3/13/95	93.91	6.09	98
5	6/19/95	93.61	6.39	91
6	9/18/95	93.36	6.64	91
7	12/18/95	93.12	6.88	91
8	3/18/96	93.08	6.92	91
9	6/17/96	92.98	7.02	91
10	9/16/96	92.89	7.11	91
11	12/16/96	92.74	7.26	91
12	3/17/97	92.72	7.28	91
13	6/16/97	92.63	7.37	91
14	9/15/97	92.55	7.45	91
15	12/15/97	92.42	7.58	91
16	3/16/98	92.42	7.58	91
17	6/15/98	92.34	7.66	91
18	9/14/98	92.28	7.72	91
19	12/14/98	92.16	7.84	91
20	3/15/99	92.17	7.83	91
Swap payment on:	6/14/99			

Note: Given these rates, the price of a $1 zero-coupon bond that matures on 6/14/99 would be .70667, and its semiannual bond equivalent yield would be 7.0658%.

Exhibit 2

Cash Consequences of a Change in a Forward Rate



date is a full 5 years away. To make the example more concrete, suppose that the forward swap's notional principal amount is $100 million. Suppose too that the fixed rate for this swap is 7.83 percent, which is the forward value of 3-month LIBOR implied by the March 1999 Eurodollar futures contract. This may not be strictly the correct thing to do, but throughout this note we use futures rates in lieu of forward rates because we have much better information about the futures rates. And, although the purpose of this note is to explain why the two rates should be different, we can use the behavior of futures rates as an excellent proxy for the behavior of forward rates.

The value of a basis point

Under the terms of this forward swap, if the value of 3-month LIBOR turns out to be 7.83 percent on March 15, 1999, no cash changes hands at all on June 14. For each basis point that 3-month LIBOR is above 7.83 percent, the person who is long the swap (that is, the person who pays fixed and receives floating) receives $2,527.78 [= (0.0001 x (91/360)) x $100,000,000] on June 14, 1999. For each basis point that 3-month LIBOR is below 7.83 percent, the person who is long the swap pays $2,527.78.

Thus, the nominal value of a basis point for this swap is $2,527.78, with the cash changing hands five years in the future.

Eurodollar futures

The futures market has based much of its success on a single operating principle. That is, all gains and losses must be settled up at the end of the day—in cash. This is as true of Eurodollar futures as it is of any futures contract.

Consider the March 1999 Eurodollar futures contract. When it expires on March 15, 1999, its final settlement price will be set equal to 100 less the spot value of 3-month LIBOR on that day. Before expiration, the Eurodollar futures price will be a function of the rate that the market expects. If there were no difference between a futures contract and a forward contract, and if the market expected a forward rate of 7.83 percent, for example, the futures price would be 92.17 [= 100.00 - 7.83]. If the market expected 7.84, the futures price would be 92.16. That is, a 1 basis point increase in value of the forward rate produces a 1-tick decrease in the futures price.

Under the Chicago Mercantile Exchange's rules, each tick or .01 in the price of a Eurodollar futures contract is worth $25. This is true whether the futures contract expires ten weeks from now, ten months from now, or ten years from now. The nominal value of a basis point change in the underlying interest rate is always $25.

Reconciling the difference in cash flow dates

We now have two cash payments that are tied to the same change in interest rates. For the particular forward swap in our example, a 1-basis-point change in the expected value of 3-month LIBOR for the period from March 15 to June 14, 1999 changes the expected value of the swap settlement on June 14 by $2,527.78. At the same time, a 1-basis-point change in the same rate produces a $25 gain or loss that the holder of a Eurodollar futures contract must settle today. The difference in timing is illustrated in Exhibit 2.

The simplest way to reconcile the timing difference is to cast the two amounts of money in terms of present values. Eurodollar futures are easy to handle. Because gains and losses are settled every day in the futures market, the present value of the $25 basis point value on a Eurodollar futures contract is always $25.

The present value of the $2,527.78 basis point value for the swap can be determined using the set of futures rates provided by a full strip of Eurodollar futures. For example, if we suppose that $1 could be invested on June 13, 1994 at the sequence of rates shown in Exhibit 1 — for example, 4.56% for the first 98 days, 5.16% for the next 91 days and so on — the total value of the investment would grow to $1.41509 by June 14, 1999. Put differently, the present value in June 1994 of $1 to be received in June 1999 would be $0.70667 [= $1 / $1.41509]. This is shown at the bottom of Exhibit 1 as the price of a zero-coupon bond with five years to maturity. At this price, the present value of $2,527.78 five years hence would be $1,786.30 [=$2,527.78 x 0.70667].

Hedging the forward swap with Eurodollar futures

Given these two present contracts values, 71.45 [= $1,786.30/$25.00] Eurodollar futures would have the same exposure to a change in the March 1999 3-month forward rate as would $100 million of the forward swap. For someone who is short the swap (that is, receiving fixed and paying floating),

Exhibit 3
Swap and Eurodollar Futures P/Ls

Interest rate changes		Short swap P/L				
Forward rate	Term rate on zero-coupon bond	Nominal value (as of 6/14/99)	Price of zero-coupon bond	Present value (as of 6/13/94)	Short Eurodollar P/L	Net
(basis points)		(notional principal amount = $100 million)			(71.45 contracts)	
10	10.3	($25,278)	0.70315	($17,774)	$17,863	$89
0	0	0	0.70667	$0	$0	$0
-10	-10.3	$25,278	0.71020	$17,952	($17,863)	$89

the appropriate hedge against a change in the forward rate would be a short position of 71.45 Eurodollar futures. Considering what has gone into this calculation, the number of Eurodollar futures needed to hedge any leg of a swap whose floating rate is 3-month LIBOR would be

Hedge Ratio =
NPA x [.0001 x Days/360] x Zero-Coupon Bond Price/$25

where *NPA* is the swap's notional principal amount, or $100 million in our example. The .0001 represents a 1-basis-point change in the forward rate. *Days* is the number of days in the period, which is 91 in our example. The *Zero-Coupon Bond Price* is the price today of a bond that pays $1 on the same day that the swap settlement is paid. In our example, the swap settlement is 5 years away, and the price of such a bond is 0.70667. The $25 is simply the present value of a basis point for a Eurodollar futures contract.

The other source of interest rate risk in the forward swap

Because any gain or loss on the swap is realized only at the end of the term, a swap can have unrealized asset value. In particular, the present value of a short position in the forward swap in our example can be written as

Swap Value =
NPA x [(X - F) x Days/360] x Zero-Coupon Bond Price

where *X* is the fixed rate at which the swap was struck originally and *F* is the current market value of the forward rate. From this we can see that the unrealized asset value of a swap depends both on the difference between the swap's fixed rate and the forward rate and on the present value of a dollar to be received on the swap's cash settlement date.

The practical importance of this expression is that there are really two sources of interest rate risk in a forward swap. The first, which we have dealt with already, is uncertainty about the forward rate, *F*. The other is uncertainty about the zero-coupon bond price, which reflects uncertainty about the entire term structure of forward rates extending from today to the swap's cash settlement date. If the forward rate is below the fixed rate, for example, the person who is receiving fixed and paying floating has an asset whose value is reduced by a general increase in interest rates. To get complete protection against interest rate risk, the swap hedger not only must offset the exposure to changes in the forward rate, but exposure to changes in the term or zero-coupon bond rate as well. The simplest way to hedge against exposure to changes in zero-coupon term rates is to buy or sell an appropriate quantity of zero-coupon bonds whose maturity matches that of the swap.

Interaction between the two sources of risk

Now we have come to the heart of the difference between a swap and a Eurodollar futures contract. With Eurodollar futures, the only source of risk is the forward or futures rate. When the futures rate changes, the holder of the futures contract collects all of the gains or pays all of the losses right away. The holder of the swap, on the other hand, faces two kinds of risk—a change in the forward rate and a change in the term rate.

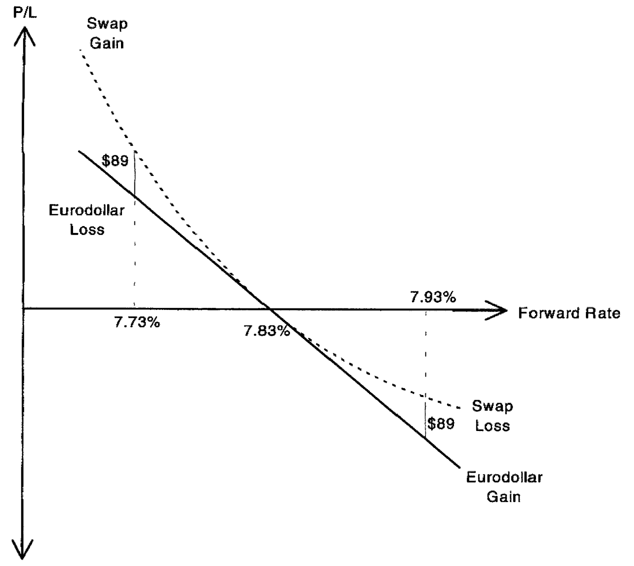
To see why this matters, consider what happens to a short swap and a short Eurodollar position if all 20 of the 3-month spot and forward rates from June 1994 through March 1999 either rise or fall by 10 basis points. The results of such an exercise are shown in Exhibit 3. Note, first, that the $17,863 gain on the short Eurodollar position when the March 1999 futures rate rises 10 basis points is the same as the $17,863 loss when the futures rate falls 10 basis points.

Similarly, the nominal loss on the short swap — $25,278 — when the March 1999 forward rate rises is equal to the nominal gain when the forward rate falls. Notice, however, that the present values of the gain and the loss on the swap are not the same. This is because the price of the zero-coupon bond falls when the forward rates rise and rises when the forward rates fall. Taking the rates in Exhibit 1 as our starting point, the price of the zero-coupon bond falls to $0.70315 per dollar when all of the forward rates increase 10 basis points. The price of the zero increases to $0.71020 when all of the forward rates fall 10 basis points. (Because of differences in compounding conventions, the semiannual bond equivalent yield on the zero-coupon bond changes by 10.3 basis points when the various forward rates change 10 basis points.)

With these changes in the price of the zero-coupon bond, the present value of the loss on the swap if all rates rise 10 basis points is $17,774 [= $25,278 x 0.70315], while the present value of the gain on the swap if rates fall 10 basis points is $17,952. As a result, we find that the short Eurodollar position makes $89 more than is lost on the swap if all forward rates rise and loses $89 less than is gained on

Exhibit 4

The Convexity Difference Between Swaps and Eurodollar Futures



the swap if interest rates fall.

A familiar way of depicting this comparison is provided in Exhibit 4. A short swap, which requires the holder to pay a floating or variable rate such as 3-month LIBOR while receiving a known fixed rate, is much the same as owning a bond that is financed with short-term money. The price/yield relationship for such a position exhibits what is known in the fixed-income trade as positive convexity. That is, the price increases more when yields fall than the price falls when yields rise. In our example, the increase in the swap's price was $17,952 while the decrease in its price was only $17,774. A Eurodollar futures position, on the other hand, exhibits no convexity at all. Each basis point change in the forward rate is worth $25 today no matter what the level of the interest rate. The short Eurodollar position makes $17,863 for a 10 basis point increase in rates and loses $17,863 for a 10 basis point decline in rates.

Because of the difference in the convexities of the two instruments, a short swap hedged with a short position in Eurodollar futures benefits from changes in the level of interest rates. As shown in Exhibit 4, the difference in convexities for the forward swap in our example is worth $89 if rates rise 10 basis points and $89 if rates fall 10 basis points.

Trading the hedge

Exhibit 4 provides an especially useful way to illustrate the nature of the trade. For example, if interest rates fall 10 basis points, the hedger of the short swap is $89 ahead of the game. At this point, the hedger could (in principle, if it weren't for the costs imposed by bid/asked spreads and brokerage) close out the position and pocket the $89. On the other hand, the hedger could view this as a vehicle for trading Eurodollar futures that would eventually accumulate a substantial amount of money. Notice that as rates fall, the number of futures needed to hedge the position increases, which requires selling the additional contracts at a higher price. On the other hand, as rates rise, the number of futures needed to hedge the position falls, which requires the hedger to cover some of the short futures by buying the excess contracts at a lower price.

HOW MUCH IS THE CONVEXITY BIAS WORTH?

The difference in the performance of a swap and the performance of a Eurodollar futures contract depends on three things:

- the size of the change in the forward rate
- the size of the change in the term rate (or zero-coupon bond price), and
- the correlation between the two.

These points are illustrated in Exhibit 5, which shows the net hedge P/L on our $100 million forward swap for a variety of different possible rate changes.

If both rates rise 5 basis points, the net P/L is $22. If both rates rise 10 basis points, the net gain is $86, or nearly four times as much. (The net gain in this instance is less than the $89 produced by the example illustrated in Exhibit 3 because the term rate in this instance has only changed by 10 basis points rather than the 10.3 basis points produced by a parallel shift in all 3-month spot and forward rates.) Also, if the forward rate rises 10 basis points while the zero-coupon rate rises only 5 basis points, the net P/L is $43. From this we can conclude that the value of the convexity difference is greater when interest rates are volatile than when they are stable.

Exhibit 5 also allows us to see the importance of correlation. The net P/Ls are positive if the two interest rates both rise or both fall. If one rate falls while the other rises, the hedged position actually loses money. If one rate changes while the other does not, there is neither a gain nor a loss.

Moreover, if the zero-coupon yield is just as likely to rise as it is to fall no matter what happens to the forward rate, the expected or average net P/L is also zero. For example, if the forward rate increases 10 basis points, the net P/L is a gain of $86 if the zero-coupon rate also increases 10 basis points. The

Exhibit 5

Net P/Ls for a Short Swap Hedged with Short Eurodollar Futures

Zero-coupon yield change (bp)	Forward rate change (bp) -10	-5	0	5	10
10	($86)	($43)	$0	$43	$86
5	($43)	($22)	$0	$22	$43
0	$0	$0	$0	$0	$0
-5	$43	$22	$0	($22)	($43)
-10	$86	$43	$0	($43)	($86)

Note: Based on Eurodollar futures rates in Exhibit 1.

net P/L is a loss of $86, though, if the zero-coupon rate falls 10 basis points. If the probability of the zero-coupon rate rising is a half no matter what happens to the forward rate, then the expected or probability weighted average gain would be zero.

How correlated are the rates?

As it happens, forward interest rates and their respective term or zero-coupon rates tend to be very highly correlated. Eurodollar futures rates and strips can be used to estimate the correlation. Exhibit 6 shows, for example, the relationship between changes in 3-month rates 4-3/4 years forward and changes in 5-year zero-coupon term rates. As you can see, the correlation is not perfect, but with only a few exceptions, increases in the forward rate are accompanied by increases in the term rate, and decreases in the forward rate are accompanied by decreases in the zero-coupon term rate.

Estimating the value of the convexity bias

To get a rough idea of how much the convexity bias might be worth, we used actual Eurodollar futures data to calculate hedge 1-week P/Ls for 3-month forward swaps with 2 years and 5 years to final cash settlement. The calculations were much like those summarized in Exhibit 3. In particular, we used 1-week changes in the price of the eighth contract in an 8-contract strip to represent the change in a 3-month forward rate 1-3/4 years forward. We used all eight rates implied by the 8-contract strip to calculate 2-year zero-coupon bond prices and then calculated the 1-week price changes associated with 1-week changes in the 2-year term rate. For the longer-dated forward swap, we used the change in the price of the twentieth contract in a 20-contract strip to represent the change in a 3-month forward rate 4-3/4 years forward and all 20 rates in the strip to calculate the price of a 5-year zero-coupon bond.

Exhibit 6

Changes in 5 Year Term Rates vs. Changes in the 4-3/4 Year Futures Rate

(weekly interval, 7/10/92 through 7/1/94)

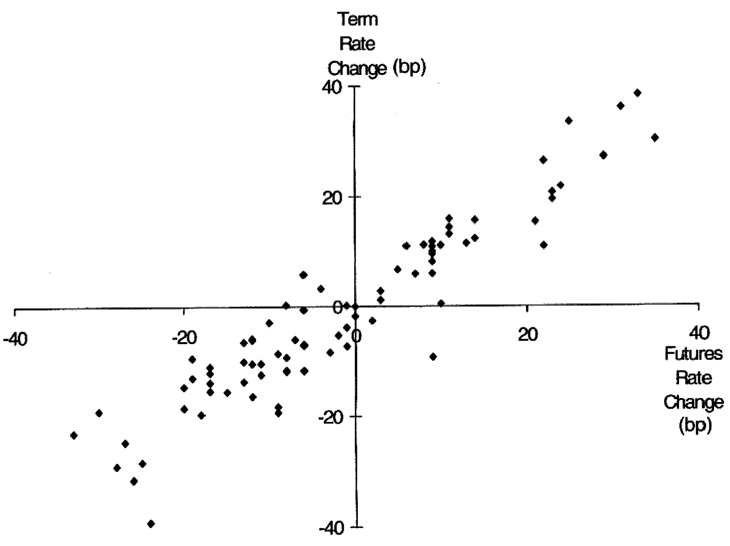
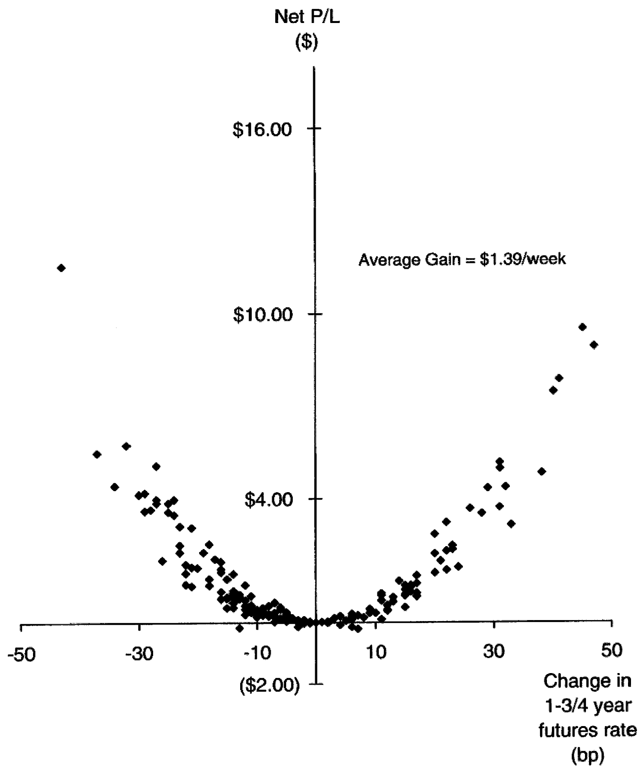


Exhibit 7

Hedge P/L for a 3-Month Swap 1-3/4 Years Forward

(weekly gains per futures contract, 1/5/90 through 7/1/94)



The results of these exercises for the 3-month swap 1-3/4 years forward are shown in Exhibit 7. The results for the 3-month swap 4-3/4 years forward are shown in Exhibit 8. In both cases, the hedge P/L has been divided by the number of futures contracts in the hedge so that the results are expressed in dollars per Eurodollar futures contract. In other words, Exhibit 7 shows the distribution of hedge P/Ls per futures for contracts that would have had 1-3/4 years to expiration, while Exhibit 8 shows the distribution of hedge P/Ls per futures for contracts that would have had 4-3/4 years to expiration.

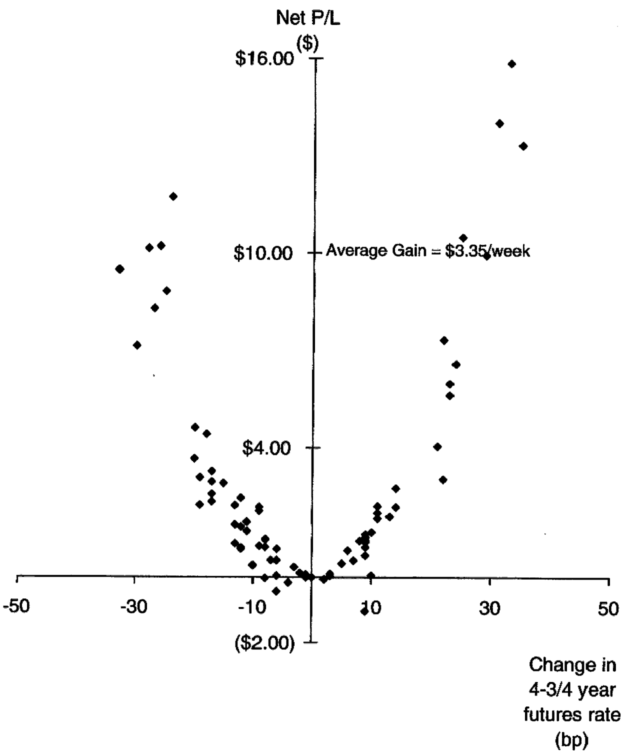
Three things stand out. First, both relationships look a lot like long straddles or strangles in Eurodollar options. In fact, while the resemblance is close, the net P/L relationships in Exhibits 7 and 8 are much more like parabolas than are straddle and strangle P/Ls. Even so, the option-like quality of a swap hedged with Eurodollar futures is pronounced.

Second, the convexity is more pronounced for the 3-month swap 4-3/4 years forward than for the 3-month swap 1-3/4 years forward. This is natural enough. Longer-dated swaps exhibit greater convexity than do shorter-dated swaps, and that is what we are seeing in these two exhibits.

Third, the distribution of outcomes looks about right. As one would expect, most of the realized outcomes involved fairly small changes in the forward rate and correspondingly

Exhibit 8

Hedge P/L for a 3-Month Swap 4-3/4 Years Forward
(weekly gains per futures contract, 7/10/92 through 7/1/94)



small net P/Ls on the hedged position. Only some of the changes were very large.

Calculating the value of the bias

Given the outcomes plotted in Exhibits 7 and 8, it is now a simple matter to calculate the average net P/L. Exhibit 7 shows that the average outcome amounted to $1.39 per Eurodollar contract per week for futures with 1-3/4 years to expiration. Exhibit 8 shows that the average hedge P/L was $3.35 per Eurodollar contract per week for futures with 4-3/4 years to expiration.

RECONCILING THE DIFFERENCE BETWEEN A SWAP AND A EURODOLLAR FUTURES CONTRACT

If you've been thinking ahead, you may see in all of this the makings of a free lunch. Exhibits 7 and 8 show all upside and no downside. As it happens, if Eurodollar futures prices were simply 100 less the appropriate forward rates, one could make money easily enough simply shorting swaps and hedging them with short Eurodollar futures. Unhappily for the swaps community, Milton Friedman was right in reminding us that there is no such thing as a free lunch — at least not for long.

If there is an advantage to being short Eurodollar futures, then one should be willing to pay for the advantage. The interesting questions then are how much this lunch should cost and how one should pay.

How one would pay for the advantage

How one pays for the advantage is comparatively easy to describe. To make the P/L distribution shown in Exhibit 7 a fair bet, the whole distribution would have to be shifted down $1.39 for the week. To make the distribution in Exhibit 8 a fair bet, the whole distribution would have to be shifted down $3.35.

The easiest way to do this is to allow the futures rate to drift down relative to the forward rate. This would cause the futures price to drift up relative to the value of the swap. At the right rate of drift, the hedger who is short the swap and short futures would expect to give up $1.39 per week or $3.35 per week due to drift but would make it back on average because of the convexity differences. In other words, the futures rate implied by any Eurodollar futures price must start out higher than its corresponding forward rate and drift down to meet it at futures contract expiration. And, for what we are doing, it makes no particular difference how one rate converges to the other. The futures rate can fall to meet the forward rate, the forward rate can rise to meet the futures rates, or the two rates can converge to one another. They are all the same to us.

If the presence of a convexity bias means that the futures rate should be higher than the forward rate, then we have to be careful about how we calculate the so-called fair value of a futures contract. The market convention is to define the fair value of the futures as 100 less the value of the forward rate. Considering the value of the bias in favor of short Eurodollar futures, the fair value of the futures contract should be lower than is provided by the conventional definition. How much lower depends on the value of the convexity bias.

Translating the advantage into basis points

In Exhibit 7 we found that the average net hedge gain for the 3-month swap 1-3/4 years forward was $1.39 per week per futures contract. At $25 per basis point for a Eurodollar contract, this means that the rate of drift for a Eurodollar futures contract with 1-3/4 years to expiration would have to be about .056 [=1.39/25] basis points per week to compensate for the convexity bias. Over the span of a quarter, the drift would have to be about .73 basis points.

In Exhibit 8, we found that the average net hedge gain for the 3-month swap 4-3/4 years forward was $3.35 per week per futures contract. Using the same arithmetic, the rate of drift for the Eurodollar contract with 4-3/4 years remaining to expiration would have to be .13 basis points per week or about 1.74 basis points per quarter.

To determine how much the difference should be between a 3-month rate 4-3/4 years forward and the 3-month interest rate implied by a Eurodollar futures contract with 4-3/4 years to expiration, the problem boils down to one of tracking a contract step-by-step and adding up the drift as the contract approaches expiration.

A WORKABLE RULE OF THUMB

There are a number of ways to determine the value of the convexity bias. One is the empirical approach illustrated in Exhibits 7 and 8. This is a perfectly good approach if one simply wants to look back and reconcile the historical differences between swaps and Eurodollar futures. The problem with this approach, however, is that it hides the assumptions that go into reckoning the value of the bias and makes it hard to adjust your estimates of the bias as your views about rate volatilities and correlations change.

Another approach is to undertake extensive and complex yield curve simulations that would allow you to estimate the cumulative gains associated with trading a hedged swap book or with financing the mark-to-market gains or losses on a futures contract. This is the approach taken in *The Financing Bias in Eurodollar Futures*, which we distributed as a research note in 1990 and which is contained in Chapter 7 of Burghardt, et. al., *Eurodollar Futures and Options: Controlling Money Market Risk* (Probus, 1991). Such interest rate simulations can produce reasonable results, but the equipment seems much too heavy for the job and may well obscure what is really going on.

The good news in this note is that the problem can be tackled with relatively light tools. The thrust of what we have done so far is that the value of the convexity bias really depends on only three things—the volatility of the forward rate, the volatility of the corresponding term rate, and the correlation between the two. As it happens, the value of the drift in the spread between the futures and forward rates that is needed to compensate for the advantage of being short Eurodollar futures can be expressed as

Drift = *standard deviation of forward rate changes*
x standard deviation of zero-coupon bond returns
x correlation of forward rate changes with zero-coupon bond returns

where *Drift* is the number of ticks that the rate spread has to fall during any given period to compensate for the convexity bias. Those who want to know where this expression comes from will find an explanation along with tips on how to apply the rule in Appendix A.

Applying the rule of thumb

Exhibit 9 provides examples of how to apply this rule to Eurodollar futures contracts with times to expiration ranging from three months to ten years. Consider, for example, the lead futures contract, which has three months remaining to expiration. The annualized standard deviation of changes in the lead futures price (or rate) is shown as 0.92% or 92 basis points. (Notice that this is an absolute and not a relative rate volatility like those quoted for Eurodollar options.) The annualized standard deviation of returns on a zero-coupon bond with an average of 4-1/2 months to maturity (the zero begins the quarter with 6 months to maturity and ends the quarter with 3 months remaining) is shown as 0.35%, or 35 basis points. This standard deviation is itself the product of the standard deviation of changes in the yield on the zero-coupon bond and the zero's time to maturity, which is also its duration. The historical correlation between these two changes is shown as .9945, which is about as highly correlated as anything can be. Taken together, we find that the required drift over a quarter of a year would be calculated as

Drift = *[0.92% x 0.35% x 0.9945]/4*
= *0.08 basis points*

In other words, for a Eurodollar futures contract with three months left to expiration, the rate of drift expressed in basis points per quarter would be 0.08 basis points. That is, the spread between the futures and forward rates would have to converge at this rate to compensate for the value of the convexity differential.

The importance of time to contract expiration

If we do the same exercise for a futures contract that has six months left to expiration, we find that the required quarterly rate of drift in the price or the rate is 0.19 basis points [= 1.03% x 0.74% x 0.9824 / 4], which is over twice as fast. The higher rate of drift is the combined effect of slightly higher rate volatilities, a very slightly lower correlation, and a very much higher duration of the zero-coupon bond.

As we saw in Exhibits 7 and 8, the value of the convexity bias depends directly on the convexity of the forward swap that is associated with the futures contract. This depends in turn on the price sensitivity of the zero-coupon bond that corresponds to the swap's maturity. Because the price of a zero-coupon bond with five years to maturity is more sensitive to a change in its yield than is the price of a zero with two years to maturity, the value of the bias is greater for a Eurodollar futures contract with 4-3/4 years to expiration than for a contract with 1-3/4 years to expiration.

The rule of thumb captures this effect nicely because the standard deviation of a zero-coupon bond's return is simply the product of the standard deviation of the zero's yield and its duration. If its yield is reckoned on a continuously compounded basis, then a zero-coupon bond's duration is simply its maturity. The result is a higher rate of drift for contracts with longer times remaining to expiration. For example, the rate of drift for a contract with five years to expiration is shown in Exhibit 9 to be about 1.5 basis points per quarter. For a contract with 10 years to expiration, the rate of drift is nearly 3 basis points per quarter.

The cumulative effect of all this drift

We know that when the futures contract expires, its final settlement price will be set equal to 100 less the spot value of 3-month LIBOR. As a result, the implied futures rate and the spot rate have to be the same at contract expiration. We also know that the implied futures rate before expiration should be drifting down relative to the corresponding forward rate so

Exhibit 9
Calculating the Value of the Convexity Bias

Years to futures expiration	Annualized standard deviations: Euro$ rate changes	Annualized standard deviations: zero yield changes*	Average years to zero maturity (avg duration)	Annualized standard deviation of zero returns	Correlation of Euro$ rate changes and zero returns**	Convexity bias (bp): per quarter	Convexity bias (bp): cumulative bias
(1)	(2)	(3)	(4) = (1) + 1/8	(5) = (3) x (4)	(6)	(7) = (2) x (5) x (6) / 4	(8)
1/4	0.92%	0.92%	3/8	0.35%	0.9945	0.08	0.08
1/2	1.03%	1.18%	5/8	0.74%	0.9824	0.19	0.27
3/4	1.12%	1.33%	7/8	1.16%	0.9726	0.32	0.59
1	1.18%	1.42%	1 1/8	1.60%	0.9646	0.45	1.04
1 1/4	1.22%	1.42%	1 3/8	1.95%	0.9581	0.57	1.61
1 1/2	1.23%	1.37%	1 5/8	2.23%	0.9527	0.65	2.26
1 3/4	1.23%	1.30%	1 7/8	2.44%	0.9484	0.71	2.97
2	1.22%	1.24%	2 1/8	2.64%	0.9448	0.76	3.73
2 1/4	1.21%	1.20%	2 3/8	2.85%	0.9419	0.81	4.54
2 1/2	1.20%	1.17%	2 5/8	3.07%	0.9396	0.86	5.40
2 3/4	1.18%	1.15%	2 7/8	3.31%	0.9377	0.92	6.32
3	1.17%	1.14%	3 1/8	3.56%	0.9363	0.98	7.30
3 1/4	1.16%	1.13%	3 3/8	3.81%	0.9352	1.04	8.34
3 1/2	1.15%	1.12%	3 5/8	4.06%	0.9344	1.09	9.43
3 3/4	1.14%	1.12%	3 7/8	4.34%	0.9339	1.16	10.59
4	1.14%	1.12%	4 1/8	4.62%	0.9336	1.23	11.82
4 1/4	1.13%	1.11%	4 3/8	4.86%	0.9335	1.28	13.10
4 1/2	1.13%	1.11%	4 5/8	5.13%	0.9336	1.35	14.45
4 3/4	1.12%	1.11%	4 7/8	5.41%	0.9339	1.42	15.87
5	1.12%	1.11%	5 1/8	5.69%	0.9342	1.49	17.36
5 1/4	1.11%	1.12%	5 3/8	6.02%	0.9348	1.57	18.93
5 1/2	1.11%	1.12%	5 5/8	6.30%	0.9354	1.64	20.57
5 3/4	1.11%	1.12%	5 7/8	6.58%	0.9361	1.71	22.28
6	1.11%	1.12%	6 1/8	6.86%	0.9369	1.79	24.07
6 1/4	1.11%	1.12%	6 3/8	7.14%	0.9378	1.86	25.93
6 1/2	1.11%	1.11%	6 5/8	7.35%	0.9388	1.92	27.85
6 3/4	1.11%	1.12%	6 7/8	7.70%	0.9398	2.01	29.86
7	1.11%	1.12%	7 1/8	7.98%	0.9409	2.08	31.94
7 1/4	1.11%	1.11%	7 3/8	8.19%	0.9420	2.14	34.08
7 1/2	1.10%	1.11%	7 5/8	8.46%	0.9432	2.21	36.29
7 3/4	1.10%	1.11%	7 7/8	8.74%	0.9444	2.27	38.56
8	1.10%	1.11%	8 1/8	9.02%	0.9457	2.34	40.90
8 1/4	1.10%	1.10%	8 3/8	9.21%	0.9470	2.39	43.29
8 1/2	1.09%	1.09%	8 5/8	9.40%	0.9484	2.44	45.73
8 3/4	1.09%	1.09%	8 7/8	9.67%	0.9497	2.51	48.24
9	1.09%	1.09%	9 1/8	9.95%	0.9512	2.57	50.81
9 1/4	1.09%	1.09%	9 3/8	10.22%	0.9526	2.64	53.45
9 1/2	1.08%	1.09%	9 5/8	10.49%	0.9540	2.71	56.16
9 3/4	1.08%	1.08%	9 7/8	10.67%	0.9555	2.75	58.91
10	1.08%	1.08%	10 1/8	10.94%	0.9570	2.82	61.73

* *Zero-coupon yield continuously compounded.*
** Equals correlation of Euro$ rates and zero-coupon yield changes.

that the two meet on contract expiration day.

The question, then, is how much different the futures and forward rates should be at any time before expiration. The answer to this question is found simply by adding up the quarterly drift estimates, which is what we have done in the last column of Exhibit 9. For example, if a futures contract with 3 months to expiration is drifting at a rate of 0.08 basis points per quarter, then the futures and forward rates would have to be 0.08 basis points apart if they are to meet exactly at expiration. On the other hand, if a futures contract with 6 months to expiration is drifting at a rate of 0.19 basis points per quarter for the first 3 months of its life and then at a rate

Exhibit 10

Standard Deviation of Eurodollar Futures Rate Changes (annualized)

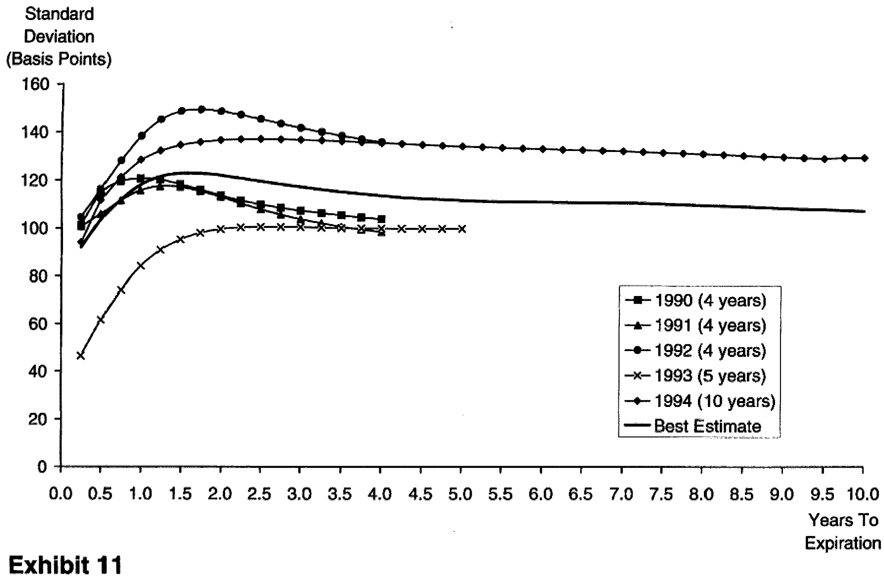
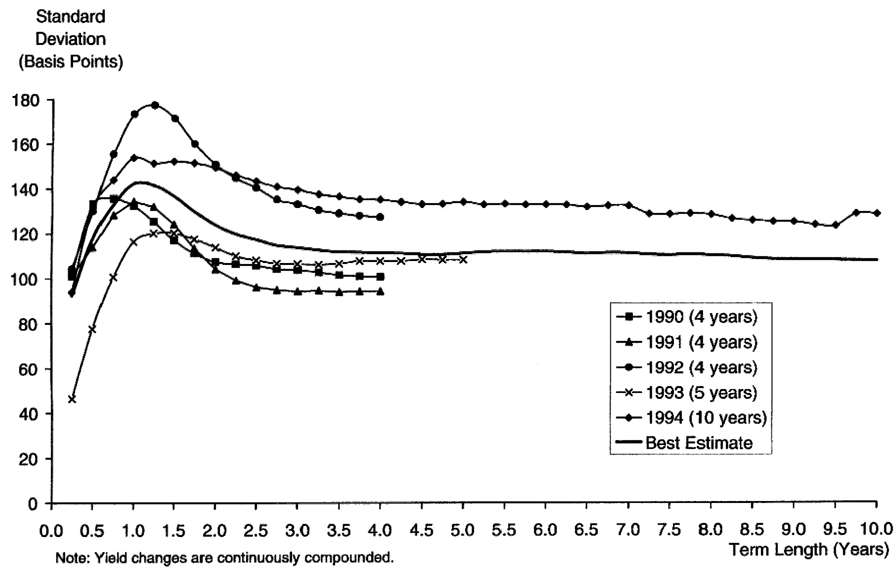


Exhibit 11

Standard Deviation of Term Yield Changes (annualized)



Note: Yield changes are continuously compounded.

of 0.08 basis points for the last 3 months of its life, the total drift in the contract's price would be 0.27 [= 0.08 + 0.19] basis points for the entire 6 months. The bias for the next contract out would be .59 [= .08 + .19 + .32] basis points, and so on down the list.

For short-dated futures contracts, all of this work adds up to comparatively little. For a contract with one year to expiration, for example, the total cumulative value of the bias adds up to only 1.04 basis points. Considering everything else that the market has to worry about, this is really nothing.

On the other hand, the adding up of these little bits of drift per quarter has a profound effect on the spread between futures and forward rates for contracts with several years to expiration. For example, the cumulative value of the bias for a contract with five years to expiration is about 17 basis points. For a contract with 10 years to expiration, the cumulative value of the bias is more than 60 basis points.

How sensitive are the estimates to the assumptions?

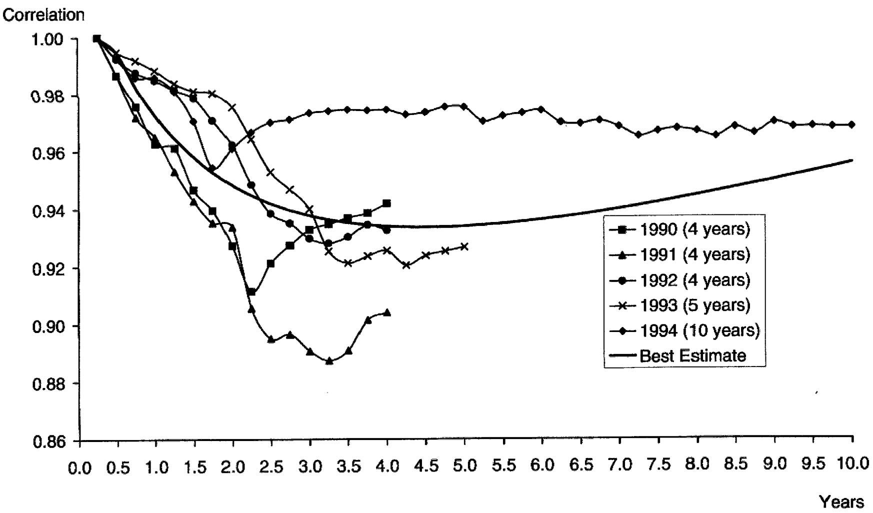
The rule of thumb makes it clear that the value of the bias is directly related to three things—the volatility of the forward rate, the volatility of the zero-coupon bond or term rate, and the correlation between the two. In particular, because the rate of drift is calculated simply by multiplying these numbers together, the required rate of drift is directly proportional to the value of each of these three things. If forward rate volatility doubles, the value of the bias doubles too. If term rate volatility doubles, the value of the bias doubles as well. If both double, the value of the bias quadruples. If both rate volatilities were increased by 10 percent, the value of the bias would be increased by 21 percent. In other words, the value that anyone places on the convexity bias depends clearly on his or her views about interest rate volatility.

To get an idea of how changeable these three key variables could be, we used Eurodollar futures data to estimate them for different time periods. The results of these exercises are shown in Exhibits 10, 11, and 12. The peculiar look of these exhibits—that is, the reason the lines have different lengths—is because the Chicago Mercantile Exchange has added futures contracts with longer times to expiration in more or less discrete chunks. For example, from 1990 to 1992, futures contracts extended out to four years, and so our estimates of rate volatilities and correlations for these years are limited to horizons of four years. By the middle of 1992, however, the CME had listed the "golds," which had five years to expiration. Then, by the end of 1993, the exchange had listed contracts with expirations extending out a full ten years.

Even with the mixed collection of data that were available to us, the results are instructive. Consider first the volatility of forward rates, which is represented by the standard deviation of Eurodollar futures rates in Exhibit 10. The annualized standard deviation of a 3-month rate four years forward in 1993 was around 100 basis points, or 1 percentage point. So far in 1994, the annualized standard deviation of a 4-year forward rate has been closer to 140 basis points. In Exhibit 9, we used 114 basis points or 1.14 percent to reckon the value of the convexity bias for a futures contract with 4

Exhibit 12

Correlation of Eurodollar Rates and Term Rates



years to expiration. (See Exhibit 9, column 2.) The estimate of 114 basis points was taken from the solid, unmarked line in Exhibit 10 that extends all the way out to ten years. This line represents our best guess about the structure of forward rate volatilities for the years 1990 through August 1994.

Because the value of the convexity bias is directly proportional to the standard deviation of forward rates and the standard deviation of term rates, the ranges of these standard deviations shown in Exhibits 10 and 11 impart substantial range to the possible value of the bias. For example, based on our best estimate of rate volatilities over the past five years, we reckoned that the value of the bias was 17 basis points for a contract with five years to expiration. Because the rule of thumb is linear in rate volatility, we can easily estimate the bias for higher or lower levels of rate standard deviations. For example, if we scale both forward and term rate standard deviations up by 15% (a reasonably high estimate given the volatility experience we saw in Exhibits 10 and 11), the bias would increase to about 22 basis points [= 17 x (1.15) x (1.15)]. On the other hand, if we scale both standard deviations down by 15% (to a low estimate), the value of the bias would decrease to about 12 basis points [= 17 x (.85) x (.85).]. So the true value of the bias for a contract with five years to expiration could easily vary between 12 and 22 basis points depending on the market's assessment of rate volatility.

Of the three key variables, the correlation between changes in forward and zero-coupon bond rates seems to be the most stable. To get a feel for these relationships, we calculated the correlations between changes in Eurodollar strips rates and changes in the rate implied by the last contract in the strip. As shown in Exhibit 12, the lowest of these correlations appear to have been in the low 90s or upper 80s, while the highest have been in the upper 90s. We used correlations in the mid 90s to construct the estimates in Exhibit 9. Given the range of correlations shown in Exhibit 12, changes in correlation from one year to the next would increase or decrease the value of the convexity bias by three or four percent, which is less than a basis point for a contract with four years to expiration and only two or three basis points for a contract with 10 years to expiration.

Practical considerations in applying the rule

One of the good things about the way we approach the problem of valuing the convexity bias is that anyone with a spreadsheet program and an understanding of rate volatilities and correlations can do the job. To do the job right, however, requires some attention to detail. For those who want to try their hand at it, follow the guidelines provided in Appendix A.

THE IMPORTANCE OF THE BIAS FOR PRICING TERM SWAPS

The swaps industry is accustomed to pricing swaps against Eurodollar futures, chiefly because Eurodollar futures prices are thought to provide the most accurate and competitive market information about forward rates. The reasoning behind such a practice is solid because the futures market is more heavily scrutinized by interest rate traders than either the cash deposit market or the over-the-counter derivatives markets.

The problem now, however, is that swaps traders are gaining a heightened appreciation for the importance of the convexity difference between swaps and Eurodollar futures. Several years ago, when futures expirations only extended out three or four years, this was not much of a problem. Today, with futures expirations extending to ten years and with longer-dated swaps trading more actively, the problem of reconciling the differences has become more acute. The effect of the convexity bias on the pricing of swaps against Eurodollar futures is illustrated for term swaps with various maturities in Exhibit 13.

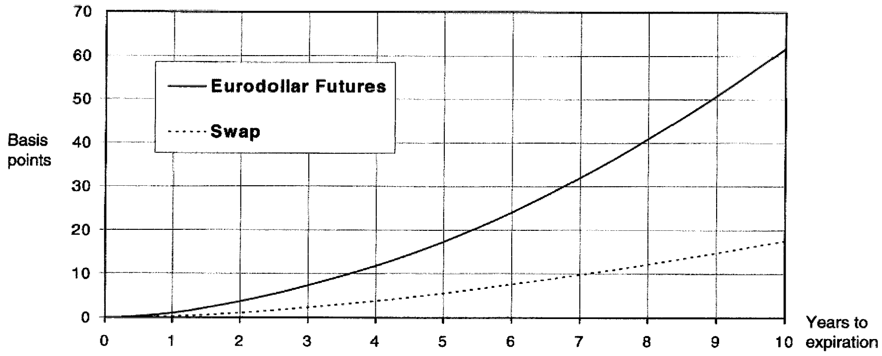
The interest rates shown in the second column represent the spot and implied Eurodollar futures rates on June 13, 1994. If we take these rates at face value and ignore the value of the convexity differences, we can calculate two kinds of term rates. One is the Eurodollar strip rate, which is the same as the rate for a zero-coupon bond with a maturity equal to the length of the strip. Another is an implied swap yield. Examples of both are shown in columns 5 and 6.

For example, the zero-coupon rate for a 5-year Eurodollar strip is shown as 7.06 percent. The swap rate next to it is 6.98 percent. The reason for the difference, which is described in Appendix B, is that a 5-year Eurodollar strip gives equal weight to all 20 of the 3-month rates that go into its calculation. An implied 5-year swap rate, however, gives greater weight to the nearby forward rates than it does to the more distant rates. As a result, if the forward rate curve slopes

[redacted]

Exhibit 13

Eurodollar and Swap Convexity Bias



Convexity-Adjusted Swap Yields

Years to expiration (1)	Eurodollar rates (MM A/360) Futures market (2)	Convexity bias (bp) (3)	Convexity adjusted (4) = (2)-(3)	Calculated term yields (SA 30/360) Eurodollar strip* (5)	Implied swap (6)	Convexity adjusted swap (7)	Swap convexity bias (bp) (8)
Spot	4.56	0.00	4.56				0.00
1/4	5.16	0.08	5.16				0.02
1/2	5.86	0.27	5.86	4.95	4.95	4.95	0.04
3/4	6.09	0.59	6.08				0.14
1	6.39	1.04	6.38	5.51	5.50	5.50	0.23
1 1/4	6.64	1.61	6.62				0.41
1 1/2	6.88	2.26	6.86	5.89	5.87	5.87	0.59
1 3/4	6.92	2.97	6.89				0.83
2	7.02	3.73	6.98	6.18	6.16	6.15	1.08
2 1/4	7.11	4.54	7.06				1.37
2 1/2	7.26	5.40	7.21	6.40	6.36	6.34	1.66
2 3/4	7.28	6.32	7.22				1.99
3	7.37	7.30	7.30	6.57	6.52	6.50	2.32
3 1/4	7.45	8.34	7.37				2.68
3 1/2	7.58	9.43	7.49	6.71	6.66	6.63	3.05
3 3/4	7.58	10.59	7.47				3.44
4	7.66	11.82	7.54	6.84	6.78	6.74	3.83
4 1/4	7.72	13.10	7.59				4.25
4 1/2	7.84	14.45	7.70	6.96	6.88	6.84	4.68
4 3/4	7.83	15.87	7.67				5.13
5	7.91	17.36	7.74	7.06	6.98	6.92	5.58
5 1/4	7.97	18.93	7.78				6.06
5 1/2	8.09	20.57	7.88	7.16	7.07	7.00	6.55
5 3/4	8.06	22.28	7.84				7.06
6	8.10	24.07	7.86	7.25	7.15	7.07	7.57
6 1/4	8.14	25.93	7.88				8.11
6 1/2	8.24	27.85	7.96	7.34	7.22	7.13	8.65
6 3/4	8.19	29.86	7.89				9.21
7	8.21	31.94	7.89	7.41	7.28	7.19	9.77
7 1/4	8.22	34.08	7.88				10.36
7 1/2	8.30	36.29	7.94	7.48	7.34	7.23	10.95
7 3/4	8.24	38.56	7.85				11.57
8	8.24	40.90	7.83	7.54	7.39	7.27	12.18
8 1/4	8.25	43.29	7.82				12.83
8 1/2	8.33	45.73	7.87	7.60	7.44	7.30	13.47
8 3/4	8.27	48.24	7.79				14.13
9	8.29	50.81	7.78	7.64	7.48	7.33	14.79
9 1/4	8.31	53.45	7.78				15.48
9 1/2	8.37	56.16	7.81	7.69	7.52	7.35	16.16
9 3/4	8.33	58.91	7.74				16.87
10	8.35	61.73	7.73	7.74	7.55	7.38	17.58

*Calculated from futures market Eurodollar rates on June 13, 1994.

Exhibit 14
Convexity Bias in Forward Swaps (bp)

Years forward	Swap term (years) 1	2	3	4	5	6	7	8	9	10
Spot	0.23	1.08	2.32	3.83	5.58	7.57	9.77	12.18	14.79	17.58
1	1.99	3.49	5.23	7.21	9.44	11.88	14.55	17.42	20.48	
2	5.11	7.05	9.25	11.71	14.39	17.32	20.46	23.78		
3	9.16	11.58	14.30	17.24	20.43	23.85	27.47			
4	14.22	17.22	20.42	23.90	27.61	31.52				
5	20.48	23.94	27.71	31.73	35.95					
6	27.70	31.81	36.14	40.69						
7	36.28	40.91	45.77							
8	45.93	51.13								
9	56.76									

(6 month LIBOR, SA 30/360)

upward, the implied swap rate is lower than the strip rate.

There is no need to take the futures rates at face value, however. If we are confident in our estimates of the value of the convexity bias, then we can adjust each of the futures rates before calculating the swap rates. No adjustment would be required for the spot rate. An adjustment of 0.08 basis points for the first of the Eurodollar futures rates is too small to have a noticeable effect. The adjustment to the rate implied by the futures contract with five years to expiration, however, is 17 basis points. As shown in columns 3 and 4 of Exhibit 13, the convexity-adjusted futures rate would be 7.74 [= 7.91 - .17] percent. Similarly, the convexity-adjusted futures rate for the longest dated futures contract, which had 10 years to expiration, would be 7.73 [= 8.35 - .62] percent to reflect an adjustment of 62 basis points.

These convexity-adjusted futures rates are a much better reflection of the forward rates implied by Eurodollar futures prices and are the rates that we use to calculate what we call convexity-adjusted implied swap rates. For example, the 5-year swap rate implied by the adjusted futures rates would be 6.92 percent, which is 6 basis points less than the 6.98 percent that one would get using the raw unadjusted rates. The 10-year swap rate would be 7.38 percent, which is 17 basis points less than the 7.55 percent obtained from the unadjusted futures rates.

In other words, if our estimates of the convexity biases are reliable, then a 5-year swap rate should be about 6 basis points lower than the rate implied by raw Eurodollar futures rates. A 10-year swap rate should be about 17 basis points lower. Put differently, if one wants to know whether swap yields are rich or cheap relative to Eurodollar futures, the convexity-adjusted yield spreads are the standards against which the market spreads should be compared.

Biases in forward swap rates

The market for forward swaps seems to have been growing recently. For example, one can find more or less active markets for 5-year swaps 5 years forward, or for 2-year swaps 8 years forward. For such swaps, the convexity bias can loom fairly large.

Exhibit 14 shows what the bias would be for a wide range of spot and forward swaps given the volatility and correlation assumptions that we have used. Along the top row, for example, are the calculations for spot swaps with terms ranging from 1 to 10 years. The numbers in this row are the same as those in the right hand column of Exhibit 13. Along the second row are the biases for swaps that begin one year in the future. For example, the value of the convexity adjustment for a five-year swap that begins one year in the future is about 9 basis points. In contrast, the value of the convexity bias for a 5-year swap that begins 5 years in the future would be about 36 basis points.

THE MARKET'S EXPERIENCE WITH THE CONVEXITY BIAS

Exhibit 15 provides an interesting look at how the market's appreciation for convexity bias has grown over the past couple of years. We used data on 5-year swap rates to calculate the spread between market swap rates and the swap rates that can be calculated using Eurodollar futures rates. The solid line represents the spread between market swap rates and the swap rates implied by convexity-adjusted futures rates. The dashed line represents the spread between market swap rates and the swap rates implied by the raw unadjusted futures rates. Notice that in 1992, the spread between the market and raw implied swap rates was around zero. In other words, in 1992, swaps appear to have been priced right on top of the Eurodollar futures rate curve. At the same time, the spread between market swap rates and the convexity-adjusted implied swap rates was around 6 basis points.

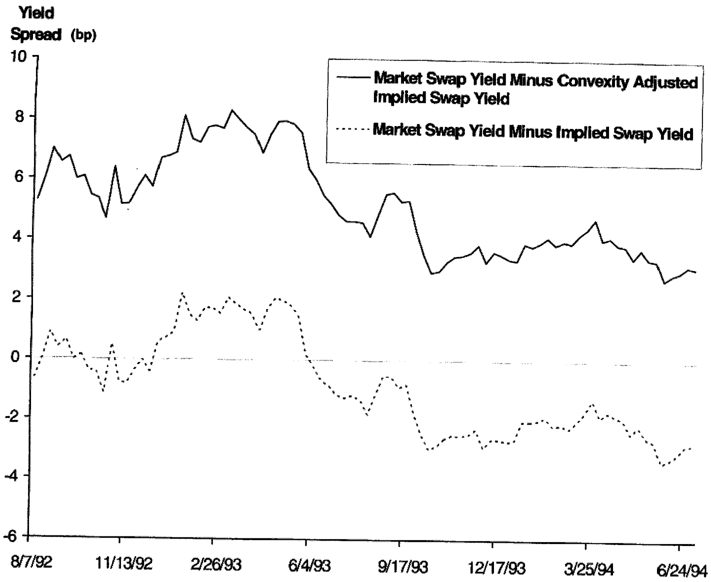
Since then, these spreads have fallen. Market swap rates now tend to trade below those that are implied by raw, unadjusted Eurodollar futures rates. At the same time, the spread between market and convexity-adjusted rates has been drifting down toward zero. In both cases, the drop in the spread suggests that the swaps market is adapting to the value of the bias in short Eurodollar futures relative to short swaps. But the adaptation appears to be incomplete. Given our estimates of the value of the convexity bias, there still seems to be some advantage in hedging a short swaps book with short Eurodollar futures.

NOW WHAT?

The natural question to ask now is what can be done with this information. Several possibilities come to mind.

Exhibit 15

Spreads Between Market and Implied Swap Yields



Note: 5 year swap rates courtesy of NationsBanc - CRT

Running a short swaps book

Given the size of the swaps market, the value of knowing how to price swaps correctly against Eurodollar futures prices is enormous. If a 5-year swap is mispriced by as little as two or three basis points against Eurodollar futures, the mispricing is worth about $80,000 on a $100 million swap. If a 10-year swap is mispriced by as little as 5 basis points, and our conversations with swaps traders suggests that this is possible, the mispricing is worth about $350,000 on a $100 million swap. These are large amounts of money and suggest that there is a lot at stake. For one thing, it suggests that a swaps desk can still make money by shorting swaps and hedging them with short Eurodollar futures.

Marking a swaps book to market

Not that bank comptrollers and risk managers need any more to worry about, but the value of the convexity bias between swaps and Eurodollar futures raises a big question about how a derivatives book should be marked to market. The standard for many banks is to mark its swaps to market using Eurodollar futures rates. This standard makes good sense because Eurodollar futures prices are the result of a much more open and competitive market process than are swap yields in the over-the-counter market. The problem we find now, however, is that Eurodollar futures prices produce forward rates that are higher than the forward rates that should be used to value swaps.

This leaves comptrollers and risk managers with a difficu choice. One approach is to stick with raw, unadjusted future rates. The advantage to this approach is that the rates are easy to calculate and to document and no one can tinker wit them. The disadvantage to this approach, though, is that the true value of the swaps book is misstated.

The other approach is to make what seems like a reasonable allowance for the value of the convexity bias. This has the advantage of providing better estimates of the value of th swaps book and of providing correct incentives for a swaps desk. The disadvantage is that convexity-adjusted Eurodollar futures rates depend so much on assumptions about rate volatilities.

Volatility arbitrage

Because the spread between swap and Eurodollar rates should depend on expected interest rate volatilities every bit as much as the prices of caps and swaptions, one should be able to detect differences in implied rate volatilities and to construct trades that profit from differences between the two markets. For example, the spread between a 5-year swap yiel and the swap yield implied by a 5-year strip of Eurodollar futures can be used to impute an expectation about interest rate volatilities from the perspective of swaps and Eurodollar traders. The price of a 5-year interest rate cap, on the other hand, reflects that market's expectations about interest rate volatilities over the same period. A sharp trading desk should be able to arbitrage differences between the two markets' implied rate volatilities.

Evaluating term TED spreads

A trade that has gained considerable popularity over the past few years has been to spread Treasury notes against strips of Eurodollar futures. In practice, the market has viewed this trade as a way of trading the yield spread between private bank paper and Treasury paper. Now we find that the rates implied by Eurodollar futures prices reflect a convexity bias, which means that these trades have a volatility component as well. For notes with five to ten years to expiration, the value of the convexity bias can loom fairly large. The imputed credit spread between the yield on a 5-year Eurodollar strip and a 5-year Treasury note really should be about 6 basis points narrower than it appears to be. In light of the comparatively tight spreads at which Eurodollars have been trading against Treasury notes anyway, such an adjustment would make the imputed credit spread appear to be paper thin rather than merely narrow.

APPENDIX A

Deriving the Rule of Thumb

The rule of thumb for calculating the rate of drift in Eurodollar rates relative to forward rates stems directly from calculating the expected gain when a forward swap is hedged with Eurodollar futures and applying the "no free lunch" principle.

Swap value

The net present value of a forward swap that receives fixed and pays floating for a 3-month period is:

$$NPV = NPA \times (X - F) \times (Days/360) \times P_z$$

where *NPA* is the swap's notional principal amount, *X* is the fixed rate at which the swap is struck, *F* is the forward rate, *Days* is the actual number of days in the swap period to which the floating and fixed rates apply, and P_z is the fractional price of a zero-coupon bond that matures on the swap payment date (which is *Days* following the swap rate setting date). The interest rates in this expression are expressed in percent (that is, 7 percent would be .07). If we multiply and divide this expression by \$1,000,000 as well as by 90, we get

$$NPV = (NPA/\$1MM) \times [(X - F) \times 10{,}000] \times [(Days/90) \times (90/360) \times \$100] \times P_z$$

which is fairly messy but allows us to arrive at

$$NPV = (NPA/\$1MM) \times (X^{*}-F^{*}) \times (Days/90) \times \$25 \times P_z$$

in which X^{*} and F^{*} are expressed in basis points. We also find the \$25, which corresponds nicely to the value of a tick or basis point on a Eurodollar futures contract. The value of *Days/90* compensates for the actual length of the swap period.

When a typical swap is transacted, we begin with $X^{*} = F^{*}$ so that the net present value of the swap is zero. When interest rates change, both F^{*} and P_z change, and both contribute to the swap's profit or loss.

Swap P/L and Hedge Ratio

For a change of ΔF^{*} in the forward rate and ΔP_z in the price of the zero, the profit on the forward swap is

$$\Delta NPV = -(NPA/\$1MM) \times (Days/90) \times \$25 \times \Delta F^{*} \times (P_z + \Delta P_z)$$

Because the change in the value of one Eurodollar futures contract is equal to $-\$25 \times \Delta F^{*}$, the number of futures contracts needed to hedge against unexpected changes in rates would be

$$Hedge\ Ratio = -(NPA/\$1MM) \times (Days/90) \times P_z$$

This hedge ratio makes sense. The minus sign indicates that the hedger must short the contracts, *NPA/\$1MM* captures the nominal number of contracts required, *Days/90* reflects the importance of the day count in the swap, and P_z provides the present value correction for the difference in timing of the cash flows on the futures and the swap.

Eurodollar P/L

Given this hedge ratio, the profit on the short Eurodollar futures position would be

$$(NPA/\$1MM) \times (Days/90) \times P_z \times (\Delta F^{*} + Drift) \times \$25$$

where Drift represents the systematic change in the Eurodollar futures rate relative to the forward rate needed to compensate for the convexity difference between the swap and the futures contract.

Expected Hedge P/L

To eliminate any possibility of a free lunch in this hedge, the expected profit of the hedged swap must be zero. Put differently the expected profit on the swap must exactly offset the expected profit on the Eurodollar position. Because the *[(NPA/\$1MM) x (Days/90) x \$25]* is common to both the profit on the swap and the profit on the Eurodollar position, this part of both expressions cancels out. The result of setting the two combined profits equal to zero and rearranging shows us that

$$E[\Delta F^{*} \times (P_z + \Delta P_z)] = E[P_z \times (\Delta F^{*} + Drift)]$$

where *E[ ]* represents the market's expectation today of whatever is contained inside the brackets. Because P_z is a known number, we can solve for the drift by dividing through by P_z within the expectations to get

$$E[Drift] = E[\Delta F^{*} \times (\Delta P_z/P_z)]$$

If we combine this expression with the fact that the average move in forward rates and term rates will be zero and use the formula for correlation, we arrive at the rule of thumb:

$$E[Drift] = stdev(\Delta F^{*}) \times stdev(\Delta P_z/P_z) \times correlation(\Delta F^{*}, \Delta P_z/P_z)$$

This rule of thumb assumes nothing, by the way, about the distribution of rate changes.

Practical considerations

- The drift is expressed in basis points per period if the standard deviation of ΔF^{*} is in basis points per period.
- To use volatilities from the options market, relative or percentage rate volatilities must be converted to absolute rate volatilities by multiplying by the level of the interest rate.
- $\Delta P_z/P_z$ is the *unexpected* return on a zero-coupon bond over the period. It should be expressed as a fraction (for example, as 0.015). The easiest way to compute the standard deviation of $\Delta P_z/P_z$ is to break it into two parts: the standard deviation of the zero's continuously compounded yield and duration. (See Appendix B for the method used to compute continuously compounded zero-coupon yields from Eurodollar futures rates
- The length of the period over which you calculate changes in rates is not terribly important as long as the duration for the zero-coupon bond is chosen to be its average years to maturity over the period. A period of one day would be theoretically correct, because mark-to-market actually occurs daily in the futures market. But this would be computational overkill. Using a quarterly period produces almost the same result as daily calculations but involves a lot less work.

APPENDIX B

Calculating Eurodollar Strip Rates and Implied Swap Rates

Eurodollar strip rates

A Eurodollar strip is a position that contains one each of the contracts in a sequence of contract months. For example, a 1-year strip might contain one each of the June '94, September '94, December '94, and March '95 contracts. A 2-year strip would contain these plus one each of the June '95, September '95, December '95, and March '96 contracts. The rates implied by a strip of Eurodollar futures prices together with an initial spot rate can be used to calculate the terminal value of $1 invested today. For example,

$$TW_T = [1 + R_0\ (D_0/360)]\ x\ [1 + F_1\ (D_1/360)]\ x \\ \ldots x[1+ F_n\ (D_n/360)]$$

where

- TW_T is the terminal value (i.e., terminal wealth) of $1 invested today for T years
- R_0 is spot LIBOR to first futures expiration
- F_1 is the lead futures rate [= 100 - lead futures price]
- F_n is the futures rate for the last contract in the strip
- D_i is the actual number of days in each period, i = 0,...,n

From this value of terminal wealth, we can calculate Eurodollar strip rates in several forms including money market, semiannual bond equivalent, and continuously compounded. All three are zero-coupon bond rates implied by a strip of Eurodollar futures prices.

Money market strip yield

The money market strip yield is the value of R_{MM} that satisfies

$$[1 + R_{MM}\ (365/360)]^N\ x\ [1 + R_{MM}\ (D_f/360)] = TW_T$$

where N is the whole number of years in the strip and D_f is the number of days in a partial year at the end of the strip.

Semiannual bond equivalent yield

The semiannual bond equivalent strip yield is the value of R_{SA} that satisfies

$$[1 + R_{SA}/2]^{2T} = TW_T$$

which provides R_{SA} as

$$R_{SA} = [TW_T^{1/(2T)} - 1]\ x\ 2$$

Continuously compounded yield

For computing returns on zero-coupon bonds, continuously compounded yields are the most convenient because the durat of a zero-coupon bond is equal to its maturity when yield chan are continuously compounded. The continuously compounded yield is the value of R_{CC} for which

$$e^{T x R_{CC}} = TW_T$$

where e is the base for natural logarithms. This can be solved

$$R_{CC} = ln\ (TW_T)/T$$

where ln( ) is the natural log.

Zero-coupon bond price

The price of a $1 par value zero-coupon bond that matures at T

$$P_z = 1/TW_T$$

Implied swap rates

A conventional fixed/floating interest rate swap typically is priced as if it contains a long position in a floating rate note an short position in a fixed-rate note. At the time of the transactio the fixed and floating rates are set so that the net present value the swap is zero. If the initial floating rate is set equal to the market rate for the term of the floater — for example, equal to 3-month LIBOR if the swap has 3-month reset dates — then on can assume the hypothetical floater would trade at par. As a re one can assume that the fixed rate on the swap must be set so the hpothetical fixed-rate note would also trade at par. The swa yield is simply the coupon rate that would accomplish this. Fo example, the swap yield for a 1-year swap with semiannual res dates would be the value of C that satisfied the following

$$[C/2\ x\ P_6] + [(C/2 + 100)\ x\ P_{12}] = 100$$

where P_6 is the price of a zero-coupon bond that matures in six months, and P_{12} is the price of a zero-coupon bond that mature 12 months. If one happens to be pricing a swap on a futures expiration date, the zero-coupon prices would be calculated as

$$P_6 = 1/\{[1 + F_1(D_1/360)]\ x\ [1 + F_2(D_2/360)]\}$$

$$P_{12} = 1/\{[1 + F_1(D_1/360)]\ x\ [1 + F_2(D_2/360)] \\ x\ [1 + F_3(D_3/360)]\ x\ [1 + F_4(D_4/360)]\}$$

and so forth. Note that F_1 and F_2 appear both in P_6 and P_{12} and and F_4 appear only in P_{12}. From this, one can see that the swap yield implied by a sequence of Eurodollar futures rates is a weighted average of these rates that gives greater weight to th nearby rates than to the more distant rates.

Yield curves and how to build them

Yield curves like bottles of wine come in all shapes and sizes, also like wine it is important to be able to distinguish between the contents of each variety. The purpose of this article is to highlight some of the differences between types of yield curve and to provide a guide to building a good estimate of the term structure of interest rates.

WHY BUILD A YIELD CURVE?

Valuing future cash flows and calculating forward interest rates are essential parts of a wide range of financial activities. Project managers, actuaries and anyone involved in the trading of derivative products, such as swaps and options, all need access to a consistent term structure of interest rates. It is therefore important to know how to build and use a yield curve which will provide the type of information required.

THE YIELD CURVE AND DISCOUNTING

One of the complexities of yield curve building is that, when we see an interest rate, we cannot do anything with it until we know the basis on which it is quoted. For example, whether the rate is annually or semi-annually compounded and whether interest calculations are done on the basis of a 360 or 365-day year will determine how it is used. The whole process becomes much simpler if we work with discount factors. A discount factor is the present value of one unit of currency to be paid at a future date. It is independent of market conventions and compounding intervals and applies to a unique period in time.

Yield curve building can thus be reduced to calculating a term structure of discount factors. Once the discount factor for a given period is known then an interest rate for that period can be derived with reference to the required market convention and compounding interval (see Figure 1), building a discount curve is therefore an essential precursor to building a yield curve.

A moment's reflection about the nature of the time value of money and interest rates will reveal a few things about the discount curve. Discount factors for future periods will always be greater than zero because money will always have some value even at a great distance in time. They will be less than one because having an amount of cash today is more valuable than having the cash

Figure 1: Discount factor

The present value of a future cash flow is the cash flow multiplied by the discount factor for the cash flow date.
The relationship between discount factor and interest rate depends on how the rate is expressed.

For simple interest and a 360-day year:

$$D = \frac{1}{\left(1 + R * \frac{T}{360}\right)}$$

D = Discount factor
R = Simple interest rate
T = Days to cash flow

For annual compounding:

$$D = \frac{1}{(1+R)^{\frac{T}{360}}}$$

R = Annually compounded interest rate.

Yield curves and how to build them

tomorrow as it can be invested overnight, ie, interest rates are not normally negative. The discount curve will slope downwards because if it did not then the present value of US$100 in one month could be less than the present value of US$100 in one month and a day. This is also consistent with forward discount factors, for periods starting and ending in the future, being less than one and forward interest rates being positive. These limitations on the shape of the discount curve will allow us to make sure that the yield curves, which we are building, are realistic.

ZERO-COUPON AND YIELD TO MATURITY

As our primary concern is going to be discount factors for all of the periods of concern, it makes sense to express the yield curve in terms of interest rates which have direct one to one relationships with specific discount factors. These are known as the 'spot' or 'zero-coupon' interest rates and a yield curve built from them as a 'zero-coupon yield curve'.

In contrast to this zero-coupon yield curve, the oldest and probably the least useful type of yield curve, is the yield to maturity curve. This is simply a graph of the yield to maturity of bonds against bond tenor. This is of little use because an ordinary bond yield on its own tells you nothing about the time value of money. The yield is just the single rate at which you can discount the bond's cash flows to make the total equal to the bond price. It is therefore a complex average of spot rates and says nothing about the shape of the yield curve over the bond's tenor. Of course, if the bond pays no coupons then its price divided by the amount of the payment at maturity will give you the discount factor for that period exactly; hence the name 'zero-coupon yield curve' for the term structure of spot rates.

It is possible to get useful information about discount factors out of the yields of coupon bonds, but to do this it is necessary to go through a fairly complex process. This is described in the section on bond yield curves below.

BOOTSTRAPPING A YIELD CURVE FROM MONEY-MARKET INSTRUMENTS

Bootstrapping is one of the standard ways of combining cash rates, short interest rate futures prices and interest rate swap rates to generate a zero-coupon yield curve. It is probably the simplest way to work out a term structure of discount factors and is popular with users of the interest rate derivatives markets. This method can be implemented easily on a spreadsheet and if linked to a live data feed will respond instantly to changes in market conditions.

The main disadvantage of using money-market instruments is that the curve built will not be fully in line with the Treasury bond yield curve. This is because the instruments used are not strictly 'risk free' and therefore trade at a spread over Treasuries.

If it is necessary to build a strictly risk free yield curve then the only data which can be used are the prices of risk free instruments such as Treasury bonds and bills. Before tackling the bond yield curve it is instructive to look at 'bootstrapping' with money-market instruments and swaps in more detail. A step by step approach to building this type of yield curve is set out below:

(a) Derive the cash rate out to the first futures 'delivery' International Monetary Market (IMM) date by interpolation from the quoted cash periods. These periods are normally overnight, one week, one, three and six months. Calculate the discount factor for this period.

(b) Assume that 100 less the price of the first futures contract will tell you what the forward rate is between the first futures date and the end of the deposit period three months (90 days)[1] later. Calculate the forward discount factor from this rate.

This calculation is reasonably accurate for periods of up to two to three years. After this the system of marking-to-market and margin payments with futures contracts means that it is necessary to make an adjustment involving the volatility of interest rates when deriving the forward rate from a futures price.

(c) Multiply the two discount factors calculated in (a) and (b) to get the discount factor for 90 days from the first futures date. Convert this to a spot rate.

(d) The second futures date will usually be 91 days from the previous date[2]. Using rates derived from the discount factors in (a) and (b) extrapolate for one day to find a rate for the second futures date and convert it to a discount factor.

(e) Repeat the process in (b) with the second futures contract and multiply the resulting discount factor by that calculated in (d). This will give you the discount factor for the period ending 90 days after the second futures date.

(f) The calculation can then be repeated with as many futures contracts as you have prices for to give a series of points on the zero-coupon yield curve. An example of this calculation with four futures contracts is given in Figure 2.

(g) To extend the curve beyond the last futures contract you can use swap rates as follows:

Consider a swap as an exchange of a floating-rate note (FRN) and fixed-rate bond. At each interest payment date the FRN will be priced at par so the fixed-side of the swap can be considered as a par value bond paying the fixed-rate as coupon.

Assuming that you have discount factors generated from futures prices for up to two years consider the three-year swap. From the expression for the price of this swap (see Figure 3) the only remaining unknown is the three-year discount factor. Because the fixed-side cash flows are known this can be calculated exactly.

Yield curves and how to build them

Figure 2

Days	Date	Cash rate (%)	Futures price	90-day forward rate (%)	90-day forward discount factor	Spot discount factor	Zero-coupon rate (%)
0	1–Mar–91					1	
7	8–Mar–91	6.25					6.25
19	20–Mar–91		94.04	5.96	0.98531875	0.996695	6.28
31	1–Apr–91	6.31					6.31
109	18–Jun–91					0.98206375	6.03
110	19–Jun–91		94.63	5.37	0.98675284	0.98190117	6.03
200	17–Sep–91					0.96889377	5.78
201	18–Sep–91		94.79	5.21	0.98714247	0.96874924	5.78
291	17–Dec–91					0.95629352	5.65
292	18–Dec–91		94.71	5.29	0.98694762	0.95615512	5.65
382	17–Mar–92					0.94367502	5.62

Figure 3: Swaps and Discount Factors

The fixed-side of the swap can be considered as a par value bond with a coupon equal to the swap cash flows. The relationship between price, cash flow and discount factor for a three-year swap is therefore:

$$100 = C_1D_1 + C_2D_2 + (100 + C_3)D_3$$

C_i = Cash flow at year i
D_i = Discount factor for year i

After you have the three-year discount factor the four-year swap will then provide the four-year discount factor and so on, until you have no more readily quoted swap prices (normally 10 years).

FILLING IN THE GAPS

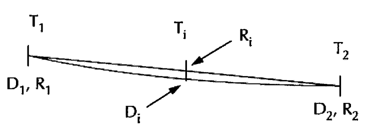
Once a set of discount factors has been generated for a discrete series of dates then cash flows on each of those dates can be valued. In order to generate a discount factor for each day between the discrete dates it is necessary to use some method of interpolation. Market practitioners generally use one of two methods.

The Straight Line Approach

Under this method an interest rate between any two points on the curve is taken to lie on a straight line between those two points. The main advantage of this method is that it is simple to calculate. Its drawbacks are that when the slope of a yield curve is changing it will give inaccurate interest rates. This is because a yield curve generated in this way will always be kinked at each data point unless it is completely linear. Also forward rates calculated from such a curve may also have undesirable characteristics and are sometimes inclined to be very irregular. It is important to note that straight line interpolation must never be used directly with discount factors (see Figure 4). The equivalent to straight line interpolation on interest rate is what is known as 'exponential interpolation on discount factor' (see Figure 5).

Figure 4: Interpolation

If interest rates follow a straight line discount factors do not.



Using rates	Using discount factor
R_1 = R2 = 6.00%	D_1 = 0.970874
T_1 = 180	D_2 = 0.943396
T_2 = 360	D_i = 0.957135
T_i = 270	R_i = 5.9713%
R_i = 6.00%	

Figure 5: Interpolation

Linear interpolation on interest rate is equivalent to exponential interpolation on discount factor.

$$D_1 = e^{-R_1T_1}, \quad D_2 = e^{-R_2T_2}$$

$$R_i = (1-\lambda)R_1 + \lambda R_2$$

$$\lambda = \frac{T_i - T_1}{T_2 - T_1}$$

$$\therefore$$

$$D_i = D_1^{(1-\lambda)\frac{T_i}{T_1}} D_2^{\lambda\frac{T_i}{T_2}}$$

USING CUBIC SPLINES

An alternative to straight line interpolation is to use some form of polynomial function to represent the yield curve. Fitting a single high order polynomial to any given set of data is unsatisfactory because there is no guarantee that the curve will not take on unrealistic forms between data points. A much more useful method of interpolation is to fit a series of linked low order polynomials to the curve. A series of linked cubic polynomials (cubic splines) is the most commonly used functional form. Because of its flexibility this technique can be used on either discount factors or interest rates.

With cubic splines the yield curve has to go through each data point as in the straight line method,

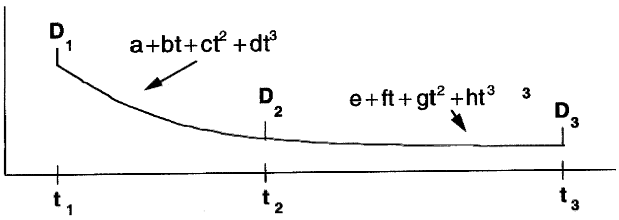
Yield curves and how to build them

but neither it, nor its slope, is allowed to be kinked at any point. The gap between each data point is represented by a unique polynomial which relates to the next one in the chain by having its slope and rate of change of slope being equal to that of its neighbour at the point at which they join. As a result of these smoothness conditions the spot and forward rate curves generated are regular and will cope with a wide range of conditions experienced in the market.

Figure 6 illustrates linking three data points (at t_1, t_2 and t_3) with two cubic splines. There are eight parameters to be determined, made up of the four coefficients from each polynomial. By having the polynomials go through each data point and by setting the first and second derivatives equal to zero where they meet, it is possible to generate six equations. In fact, for any n unknown coefficients more than two splines will give n–2 equations. The extra two equations necessary to solve for the coefficients are normally produced by setting the second derivative of the polynomials at either end of the yield curve to zero. This results in what is known as a 'neutral' cubic spline.

Figure 6: Cubic spline interpolation

Each section of the yield curve is a cubic polynomial



$D_1 = a + bt_1 + ct_1^2 + dt_1^3$

$D_2 = a + bt_2 + ct_2^2 + dt_2^3$

$D_2 = e + ft_2 + gt_2^2 + ht_2^3$

$D_3 = e + ft_3 + gt_3^2 + ht_3^3$

1st & 2nd derivatives are equal where curves join:

$b + 2ct_2 + 3dt_2^2 = f + 2gt_2 + 3ht_2^2$ $\qquad$ $2c + 6dt_2 = 2g + 6ht_2$

Two extra equations are needed so set change in slope at the ends = 0

$2c + 6dt_2 = 0$ $\qquad$ $2g + 6ht_2 = 0$

BUILDING YIELD CURVES FROM BOND DATA

The bootstrapping technique illustrated above is fine if you have enough data of the right sort. If you do not then it is necessary to resort to more complex methods of yield curve building. One of the reasons for not having enough data is the need to build a genuinely risk free term structure of interest rates. This will require the use of instruments which have no significant credit risk, such as Treasury bonds or bills.

We would, of course, like to build our risk free yield curve by the simplest method possible. This would undoubtedly be by bootstrapping, using bonds in place of swaps in the example above. Unfortunately, unless there is a series of bonds with overlapping coupon payment dates and regular maturities, as with swaps, this will not be possible. In the US, where 'on the run' Treasuries provide the necessary regularity of dates, it is possible to use this technique, but elsewhere things are not so easy. In most bond markets coupon dates do not overlap, so there are generally many more payment dates than bonds in the market and bootstrapping does not work.

In order to get over this problem of more dates than bonds it is necessary to represent the discount curve by some form of mathematical function and then estimate the parameters of the function from the bond prices. As long as there are less function parameters than bond prices this can be done without difficulty by using a technique such as multiple regression.

REPRESENTING THE DISCOUNT CURVE BY USING BASIS SPLINES

One of the most useful ways of representing the discount curve is as a weighted sum of well defined functions. Choosing the type of function to be used is quite important. The early researchers in this area used cubic polynomials as above, but more recently it has been shown that a series of what are known as 'basis splines' have better properties. The problem with cubic polynomials is that when they are used with a technique such as multiple regression it is not easy to define the accuracy of the regression results. This is because these functions introduce uncertainty due to the linkages between each segment of the curve (multicollinearity).

The reason that basis splines are useful is that they go to zero at defined points and therefore avoid the linkage problem. Basis splines can have different degrees of smoothness depending on their

Figure 7: Using basis splines

Discount factors can be expressed as a sum of weighted spline functions:

$$D(t) = \sum_{l=1}^{L} a_l f_l(t)$$

Bond prices are the sum of discounted bond cash flows:

$$P_i = \sum_{j=1}^{n} C_i \sum_{l=1}^{L} a_l f_l(t)$$

Prices can be expressed in terms of the (unknown) function weights and the (known) cashflow-function products. Weights are then determined by regression:

$$P_i = \sum_{L=1}^{L} a_l \sum_{j=1}^{n} C_{ij} f_l(t)$$

f_l = Spline function l
a_l = Function weight
$D_{(t)}$ = Discount factor
P_i = Price of bond i
C_{ij} = Cash flow of bond i at time j

Yield curves and how to build them

Figure 8: Third order basis splines

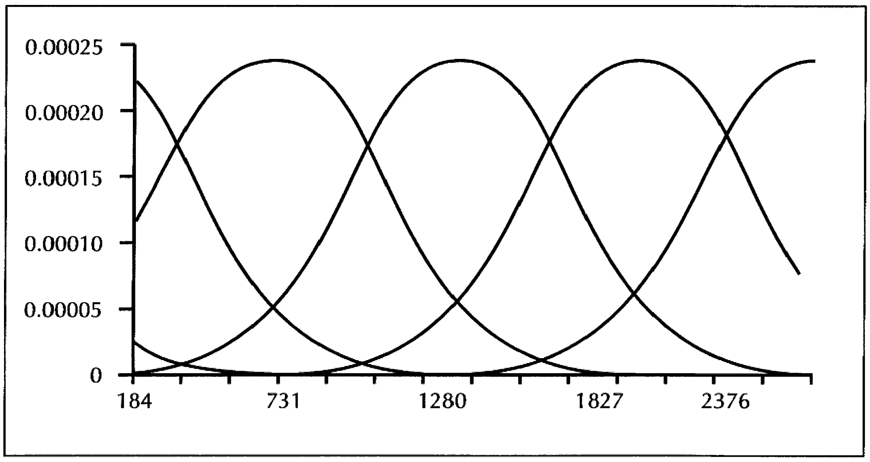
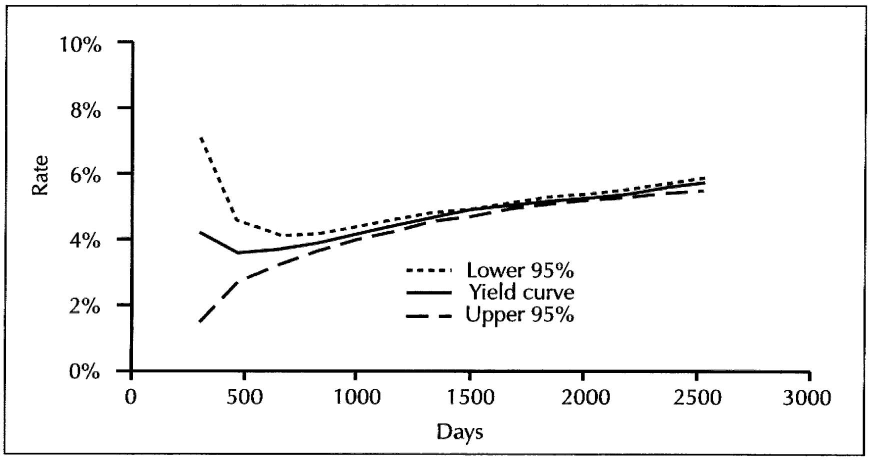


Figure 9: Confidence intervals (spline coefficient)



order. Third order basis splines are suitably smooth for yield curve building. It is not possible neatly to write down a formula representing a third order basis spline, these functions are generally built up in a step by step fashion from lower order splines (third order from second order, second from first etc).

Figure 8 illustrates a series of third order basis splines which could be used to build a yield curve spanning a period of 2741 days. Each spline function has a defined range outside of which it is zero. Notice how some of the functions start before the period of concern and others end after the period. The points at which the splines go to zero are known as 'knot points'. Note that inside the period of the yield curve one spline ends when another starts.

Figure 8 shows each spline with a weighting of one. For the sum of these functions to represent a realistic discount function each one will have a different weight.

Once the spline function knot points have been chosen the weights can be found by using multiple regression. Because each bond price in the market can be represented by the sum of the discounted coupon payments and the discount factor at each coupon date can be defined in terms of the spline functions and weights it is possible for the bond prices to be expressed in terms of the unknown function weights and the product of cash flows and spline function values, both of which are known (see Figure 7). The regression performed, therefore, uses the bond prices as dependent variables and the function cash flow products as independent variables . The function weights are the parameters being estimated.

POINTS TO WATCH

Although this technique of using basis splines is one of the best methods of yield curve building it has a number of potential drawbacks. The first of these is that the shape of the yield curve is dependent on the positioning of the knot points. For a robust curve it is necessary to make sure that there are an even number of bonds maturing between each knot point. The second is that due to limited data at the long end of a yield curve forward rates at the end of the curve can tend to be unstable, taking on unreasonably large or small values.

An example of a yield curve built using the basis spline technique is shown in Figure 9. The two lines above and below the central plot of zero-coupon interest rate against time represent rates derived from spline function weights at the 95% confidence intervals produced by the regression. Notice that, due to the lack of data, these are very wide at the short end of the curve. They are also starting to widen at the long end.

David Cox
January 1995

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NOTES

1 This period is 90 days due to the fixed-tick value on a short Euro-currency futures contract (US$25, DM25 etc) and the fact that interest is calculated on an actual over 360-day basis. For currencies where this is not the case, such as sterling, an adjustment will have to be made.

2 It is, however, occasionally 84 or 98 days later. In these cases the rate for the next futures date will have to be interpolated or extrapolated for the appropriate period.